The Dynamics of Classifying Geometric Structures

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Virginia Topology Conference 2016
Mapping Class Groups and Low Dimensional Topology
University of Virginia
Sunday 20 November 2016
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Coordinate atlases and development

Geometry: Homogeneous space $X = G / H$.

Topology: Topological manifold $\Sigma$ with universal covering $\tilde{\Sigma} \to \Sigma$ and fundamental group $\pi$.

Marking: Homeomorphism $\Sigma \to M$; the geometry on $M$ will vary, but the topology of $\Sigma$ remains fixed.

Patches $U \subset M$; Coordinate atlas of charts $U \to X$ defining local coordinates on $U$ modeled on $X$.

On overlapping patches the change of coordinates are restrictions of transformations of $X$ lying in $G$.

Charts globalize to immersion $\tilde{\Sigma} \to X$, equivariant respecting the holonomy homomorphism $\pi \to G$.

Holonomy globalizes coordinate changes.

$M(G, X)$-manifold, $(M, f)$ marked $(G, X)$-structure on $\Sigma$. 
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Ehresmann-Weil-Thurston principle

Construct a deformation space of marked \((G, X)\)-structures on \(\Sigma\) up to appropriate equivalence relation.

Holonomy defines a mapping

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\text{Def}(G, X)(\Sigma) \rightarrow \text{Hom}(\pi_1(\Sigma), G) / \text{Inn}(G)
\]

Best cases (e.g. hyperbolic manifolds): stratify into smooth manifolds and \(H\) local diffeomorphism.

Changing the marking corresponds to an action of the mapping class group

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on \(\text{Rep}(\pi, G)\) whose orbit structure defines the moduli space of \((G, X)\)-structures on \(\Sigma\).
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Example of trivial (proper) dynamics: Hyperbolic surfaces

Suppose $X = H^2$ and $G = \text{Isom}(H^2) \cong \text{PGL}(2, \mathbb{R})$.

Then $\text{Def}(G, X)(\Sigma)$ is the Fricke space $F(\Sigma)$, which identifies with the Teichmüller space $T\Sigma$ by the uniformization theorem.

$H$ embeds $F(\Sigma)$ as a connected component of $\text{Rep}(\pi, G)$:

Trivial dynamics: Action of $\text{Mod}$ on $F(\Sigma)$ is proper. Its quotient is the Riemann moduli space $M\Sigma$ of smooth Riemann surfaces of fixed topology.

For $\Sigma = T^2$, the deformation space of unit-area Euclidean structures is the upper half-plane $H^2$ with action the modular group $\text{Mod}(\Sigma) \cong \text{GL}(2, \mathbb{Z})$ acting properly by linear fractional transformations.
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In contrast, the deformation space of complete affine structures on $T^2$ is homeomorphic to $\mathbb{R}^2$, with the Euclidean structures corresponding to the origin. (O. Baues 2000)

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This chaotic action admits no reasonable quotient.

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Symplectic/Poisson structure

When adjoint representation of $G$ is orthogonal (e.g. if $G$ is reductive), then $\text{Rep}(\pi, G)$ admits a $\text{Mod}(\Sigma)$-invariant symplectic structure extending:

- Weil-Petersson Kähler form on Teichmüller component for $G = \text{SL}(2, \mathbb{R})$;
- Narsimhan-Atiyah-Bott Kähler form for $G = \text{SU}(2)$.

(Narasimhan-Seshadri moduli space of semistable bundles.)

When $\partial \Sigma \neq \emptyset$, then $\text{Rep}(\pi, G)$ inherits a Poisson structure with restriction mapping $\text{Rep}(\pi, G) \rightarrow \text{Rep}(\pi_1(\partial \Sigma), G)$ as universal Casimir. The level sets (relative character varieties) are its symplectic leaves.
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When $G$ is compact and then the components of $\text{Rep}(\pi, G)$ are indexed by $\alpha \in \pi_1([G, G])$.

Mod(Σ)-action ergodic on each component $\text{Rep}(\pi, G)$ with respect to the symplectic measure. (G-, Pickrell-Xia)

(Palesi) Ergodicity for closed nonorientable surfaces of $\chi \leq -2$ and $G = SU(2)$.

Weak-mixing: Only finite-dimensional Mod(Σ)-invariant subspaces on $L^2(\text{Rep}(\pi, G))$ are locally constant functions (trivial subrepresentations).

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  \[ \text{Rep}(\pi, G) \times \text{Rep}(\pi, G) = \text{Rep}(\pi, G \times G), \]
Continuous version of Mapping Class Group Action

Replace dynamics of action of discrete group $\text{Mod}(\Sigma)$ on $\text{Rep}(\pi, G)$ by continuous dynamical object, a family of moduli spaces, over Teichmüller space $\mathcal{T}_\Sigma$.

(Narasimhan-Seshadri 1965, . . . , Hitchin 1987): When $\Sigma \to M$ is a marked Riemann surface, then the symplectic object $\text{Rep}(\pi, G)$ inherits richer structure corresponding to reducing of structure group of $\text{Rep}(\pi, G)$ to maximal compact of symplectic group.
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- Flat $\text{Rep}(\pi, G)$- bundle $\mathcal{E}_\Sigma(G)$ over $\mathcal{M}_\Sigma$ parametrizes these structures as the Riemann surface $M$ varies:

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- Dynamics of $\mathcal{F}$ equivalent to dynamics of action of discrete group $\text{Mod}(\Sigma)$. 
Extending Teichmüller flow

Now replace dynamics of \( \text{Mod}(\Sigma) \) on \( \text{Rep}(\pi, G) \) by a measure-preserving flow \( \Phi \) on flat bundle \( U_E \Sigma(G) \).

Teichmüller unit sphere bundle \( U_M \Sigma \) over \( M \Sigma \) with Teichmüller geodesic flow \( \phi : U_M \Sigma := (U_T \Sigma) / \text{Mod}(\Sigma) \) invariantly stratified by strata \( U_M \sigma \Sigma \), indexed by partitions \( \sigma \) of \( 4g - 4 \), corresponding to zeroes of quadratic differentials.

Insert \( \text{Rep}(\pi, G) \) as the fiber:
\[
U_{E \sigma} := (U_T \Sigma) \times \text{Rep}(\pi, G) \alpha / \text{Mod}(\Sigma)
\]

Horizontally lift \( \phi \) to flow \( \Phi \) on \( U_{E \sigma} \).

Teichmüller dynamics on \( U_M \Sigma \) extends to \( (U_{E \sigma}, \Phi) \).

\( \text{Mod}(\Sigma) \)-dynamics on \( \text{Rep}(\pi, G) \) replaced by equivalent action of more tractable group \( \mathbb{R} \) (or even \( \text{SL}(2, \mathbb{R}) \)).

(Forni – G) When \( G \) is compact, then the flow \( \Phi \) is mixing (and thus ergodic) on \( U_{E \sigma} \).
Extending Teichmüller flow

- Now replace dynamics of $\text{Mod}(\Sigma)$ on $\text{Rep}(\pi, G)$ by a measure-preserving flow $\Phi$ on flat bundle $U\mathcal{C}_\Sigma(G)$. 

- Horizontally lift $\phi$ to flow $\Phi$ on $U\mathcal{C}_\Sigma(G)$.

- Teichmüller dynamics on $U\mathcal{M}_\Sigma$ extends to $(U\mathcal{C}_\Sigma(G), \Phi)$.

- $\text{Mod}(\Sigma)$-dynamics on $\text{Rep}(\pi, G)$ replaced by equivalent action of more tractable group $R$ (or even $\text{SL}(2, \mathbb{R})$).

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M_\Sigma := T_\Sigma / \text{Mod}(\Sigma)
\]

invariantly stratified by strata \( U M_\Sigma^\sigma \), indexed by partitions \( \sigma \) of \( 4g - 4 \), corresponding to zeroes of quadratic differentials.
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  - Insert $\text{Rep}(\pi, G)^\alpha$ as the fiber:

$$U\mathcal{E}_\Sigma(G)^{\sigma, \alpha} := ((U\mathcal{T}_\Sigma)^\sigma \times \text{Rep}(\pi, G)^\alpha) / \text{Mod}(\Sigma)$$
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Character functions and Hamiltonian twist flows

Elements $\gamma \in \pi_1(\Sigma)$ define character functions on $\text{Rep}(\pi, G)$:

$$\text{Rep}(\pi, G) \xrightarrow{f_{\gamma}} \mathbb{R}, \quad \rho \mapsto \mathbb{R} (\text{Tr} \rho(\gamma))$$

with Hamiltonian vector fields $\text{Ham}(f_{\gamma})$.

For the Fricke-Teichmüller component when $G = \text{PSL}(2, \mathbb{R})$, and $\gamma$ corresponding to a simple loop, $\text{Ham}(f_{\gamma})$ generates the Fenchel-Nielsen twist flows, reparametrized (Wolpert 1982).

$\gamma$ determines an oriented cycle on $\Sigma$ and the Killing vector field generating the holonomy $\rho(\gamma)$ defines a coefficient in the Lie algebra $\text{sl}(2, \mathbb{R})$, giving an infinitesimal deformation $\rho$ in $T[\rho] \text{Hom}(\pi_1(\Sigma), G)/G \sim = H_1(\Sigma, \text{Ad} \rho)$.

This deformation is supported on the cycle $\gamma$. 
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Hamiltonian flows and Dehn twists

- Dehn twist $\text{Tw}_\gamma$ generates lattice inside $\mathbb{R}$-action corresponding to $\text{Ham}(f_\gamma)$-orbits.
- $\rho(\gamma)$ elliptic element of $G = \text{SL}(2,\mathbb{R})$ $\Rightarrow$ Integral curves of $\text{Ham}(f_\gamma)$ are circles $S_{\gamma\rho}$.
- For almost every value of $f_\gamma$, the Dehn twist $\text{Tw}_\gamma$ defines an ergodic translation of $S_{\gamma\rho}$.
- Ergodic decomposition: Every $\text{Tw}_\gamma$-invariant function is a.e. $\text{Ham}(f_\gamma)$-invariant.
- For $\text{SL}(2)$, a family of simple curves exist so that $f_\gamma$ generate the coordinate ring of $\text{Rep}(\pi, G)$.
- Flows of $\text{Ham}(f_\gamma)$ generate transitive action on each connected component of where the vector fields span.
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Surfaces with $\pi \cong F_2$
Vogt-Fricke theorem and $F_2$

Let $F_2 = \langle X, Y \rangle$ be free of rank two. Then $\text{Hom}(F_2, \text{SL}(2)) \cong \text{SL}(2) \times \text{SL}(2)$ and $\text{Rep}(F_2, \text{SL}(2))$ is its quotient under $\text{Inn}(\text{SL}(2))$.

The $\text{Inn}(\text{SL}(2))$-invariant mapping $\text{Hom}(F_2, \text{SL}(2)) \rightarrow \mathbb{C}^3 \rho \mapsto \begin{bmatrix} \xi := \text{Tr}(\rho(X)) \\ \eta := \text{Tr}(\rho(Y)) \\ \zeta := \text{Tr}(\rho(XY)) \end{bmatrix}$ defines an isomorphism $\text{Rep}(F_2, \text{SL}(2)) \cong \mathbb{C}^3$. 
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Polynomial automorphisms
 Polynomial automorphisms

- Out($F_2$)-invariant commutator trace function:

$$\text{Rep}(F_2, \text{SL}(2)) \cong \mathbb{C}^3 \xrightarrow{\kappa} \mathbb{C}$$

$$(\xi, \eta, \zeta) \mapsto \xi^2 + \eta^2 + \zeta^2 - \xi\eta\zeta - 2$$

$$= \text{Tr}[\rho(X), \rho(Y)]$$
Polynomial automorphisms

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- Casimir ($\partial$-trace) for one-holed torus $\Sigma_{1,1}$. 

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▶ Nonlinear automorphisms generated by Vieta involutions:

\[
\begin{bmatrix}
\xi \\
\eta \\
\zeta
\end{bmatrix} \mapsto \begin{bmatrix}
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\eta \\
\zeta
\end{bmatrix}, \quad \begin{bmatrix}
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$$\begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \mapsto \begin{bmatrix} \eta \zeta - \xi \\ \eta \\ \zeta \end{bmatrix}, \quad \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \mapsto \begin{bmatrix} \xi \\ \xi \zeta - \eta \\ \zeta \end{bmatrix}, \quad \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \mapsto \begin{bmatrix} \xi \\ \eta \zeta - \zeta \end{bmatrix}$$

▶ Coordinate projections are double Galois coverings
 Polynomial automorphisms

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\text{Rep}(F_2, \text{SL}(2)) \cong \mathbb{C}^3 \xrightarrow{\kappa} \mathbb{C} \\
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\zeta \\
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\xi \\
\eta \\
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\end{bmatrix}
\]

- Coordinate projections are double Galois coverings
- Vieta involutions are deck transformations.
Cayley cubic $\xi^2 + \eta^2 + \zeta^2 - \xi \eta \zeta = 4$
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$$(a, b) \mapsto (a^{-1}, b^{-1}).$$

$\xi = a + a^{-1}, \quad \eta = b + b^{-1}, \quad \zeta = ab + (ab)^{-1}$
Cayley cubic $\xi^2 + \eta^2 + \zeta^2 - \xi\eta\zeta = 4$

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    $$(a, b) \mapsto (a^{-1}, b^{-1}).$$
    $$\xi = a + a^{-1}, \quad \eta = b + b^{-1}, \quad \zeta = ab + (ab)^{-1}$$
  - Homogeneous dynamics: $\text{GL}(2, \mathbb{Z})$-action on $(\mathbb{C}^* \times \mathbb{C}^*)/(\mathbb{Z}/2)$. 
$\mathbb{R}$-points: Unitary representations

Characters in $[-2, 2]$ with $\kappa \leq 2 \leftrightarrow$ SU(2)-representations.
\( \mathbb{R} \)-points: Unitary representations

- \( \mathbb{R} \)-points correspond to representations into \( \mathbb{R} \)-forms of \( SL(2) \): either \( SL(2, \mathbb{R}) \) or \( SU(2) \).
**R-points: Unitary representations**

- R-points correspond to representations into R-forms of SL(2): either $\text{SL}(2, \mathbb{R})$ or $\text{SU}(2)$.
- Characters in $[-2, 2]^3$ with $\kappa \leq 2 \leftrightarrow \text{SU}(2)$-representations.
$\mathbb{R}$-points: Hyperbolic structures on orientable surfaces

Real characters, that is when $(\xi, \eta, \zeta) \in \mathbb{R}^3$, may correspond to hyperbolic structures on the three-holed sphere $\Sigma_{0,3}$ and the one-holed torus $\Sigma_{1,1}$.

Hyperbolic three-holed spheres are parametrized by the lengths $\ell_X, \ell_Y, \ell_Z$ of $\partial \Sigma$:

$$\begin{align*}
\xi &= -2 \cosh \left( \frac{\ell_X}{2} \right) \\
\eta &= -2 \cosh \left( \frac{\ell_Y}{2} \right) \\
\zeta &= -2 \cosh \left( \frac{\ell_Z}{2} \right)
\end{align*}$$

comprising the subset $(-\infty, -2\sqrt{3}] \subset \mathbb{R}^3$.

Necessarily $k = \kappa(\xi, \eta, \zeta) \geq 18$. 
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Hyperbolic three-holed spheres are parametrized by the lengths $\ell_X, \ell_Y, \ell_Z$ of $\partial \Sigma$:

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\xi &:= -2 \cosh \left( \ell_X / 2 \right) \\
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Hyperbolic structures on one-holed tori

\(\mathbb{R}\)-points: Hyperbolic structures on one-holed tori

\[ \text{Hyperbolic structures on } \Sigma_{1,1} \text{ correspond to real characters } (\xi, \eta, \zeta) \in \mathbb{R}^3 \text{ with commutator trace } \kappa := \kappa(\xi, \eta, \zeta) < -2 \text{ corresponding to the boundary length:} \]

\[ \kappa = -2 \cosh\left(\frac{\ell_{\partial \Sigma}}{2}\right) \]

The level set \(\mathbb{R}^3 \cap \kappa = 1(\kappa)\) corresponds to hyperbolic structures on a once-punctured torus, that is, the end of \(\Sigma\) corresponding to \(\partial \Sigma\) is a cusp.

Level sets \(\mathbb{R}^3 \cap \kappa = t(\kappa)\) where \(-2 < \kappa < 2\) correspond to hyperbolic tori with one cone point of angle \(\theta\):

\[ \kappa = -2 \cos\left(\frac{\theta}{2}\right) \]

Generalized Fricke space \(F'(\Sigma)\) comprises hyperbolic structures on \(\Sigma\) with funnels, cusps or discs containing cone points.
Hyperbolic structures on $\Sigma_{1,1}$ correspond to real characters $(\xi, \eta, \zeta) \in \mathbb{R}^3$ with commutator trace $k := \kappa(\xi, \eta, \zeta) < -2$ corresponding to the boundary length:

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The level set $\mathbb{R}^3 \cap \kappa^{-1}(-2)$ corresponds to hyperbolic structures on a once-punctured torus, that is, the end of $\Sigma$ corresponding to $\partial \Sigma$ is a cusp.
**R-points: Hyperbolic structures on one-holed tori**

- Hyperbolic structures on $\Sigma_{1,1}$ correspond to real characters $(\xi, \eta, \zeta) \in \mathbb{R}^3$ with commutator trace $k := \kappa(\xi, \eta, \zeta) < -2$ corresponding to the boundary length:

  $$k = -2 \cosh \left( \frac{\ell_{\partial \Sigma}}{2} \right)$$

- The level set $\mathbb{R}^3 \cap \kappa^{-1}(-2)$ corresponds to hyperbolic structures on a once-punctured torus, that is, the end of $\Sigma$ corresponding to $\partial \Sigma$ is a *cusp*.

- Level sets $\mathbb{R}^3 \cap \kappa^{-1}(k)$ where $-2 < k < 2$ correspond to hyperbolic tori with one *cone point of angle* $\theta$:

  $$k = -2 \cos \left( \frac{\theta}{2} \right)$$
Hyperbolic structures on $\Sigma_{1,1}$ correspond to real characters $(\xi, \eta, \zeta) \in \mathbb{R}^3$ with commutator trace $k := \kappa(\xi, \eta, \zeta) < -2$ corresponding to the boundary length:

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Generalized Fricke space $\mathfrak{F}'(\Sigma)$ comprises hyperbolic structures on $\Sigma$ with funnels, cusps or discs containing cone points.
Example: The Markoff surface $x^2 + y^2 + z^2 = xyz$

$\mathbb{R}^3 \cap \kappa^{-1}(-2)$ parametrizes hyperbolic structures on the punctured torus. The origin $(0, 0, 0)$ corresponds to the unique $SU(2)$-representation with $k = -2$. The famous Markoff triples correspond to triply symmetric hyperbolic punctured tori.
Fricke orbits define wandering domains for $k > 2$

For $k \leq 18$, action is ergodic.

For $k > 18$, action is ergodic on complement of Fricke orbit.
Fricke orbits define wandering domains for $k > 2$

- Homotopy equivalences $\Sigma_{1,1} \to \Sigma_{0,3}$ define embeddings of Fricke spaces $\mathcal{F}(\Sigma_{0,3})$ in $\kappa^{-1}(k)$ for $k > 18$;
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- For $k > 18$, action is ergodic on complement of Fricke orbit.
Relative character variety for one-holed Klein bottle $C_{1,1}$

Let $k > 2$ be the commutator trace. The relative character variety is defined by:

$$-x^2 - y^2 + z^2 + xyz = k + 2$$

Each component projects diffeomorphically to the $(x, y)$-plane.
The Generalized Fricke space \( \mathcal{F}'(\mathcal{C}_{1,1}) \) of \( \mathcal{C}_{1,1} \) identifies with the subset defined by \( z > 2 \) and \( Q_z(x, y) = x^2 + y^2 - zxy < 0 \).

The trace function \( z \) corresponding to two-sided interior curve \( Z \).

The boundary trace is:

\[
\delta := Q_z(x, y) + 2 = z^2 - k =
\begin{cases}
-2 \cosh(\ell/2) & \text{for a funnel with closed geodesic of length } \ell; \\
-2 & \text{for a cusp}; \\
-2 \cos(\theta/2) & \text{for a point with cone angle } \theta.
\end{cases}
\]

Goldman – McShane – Stantchev – Ser Peow Tan

Automorphisms of two-generator free groups and spaces of isometric actions on the hyperbolic plane, DG.1509.03790
Structures on $C_{1,1}$

- The Generalized Fricke space $\mathcal{F}'(C_{1,1})$ of $C_{1,1}$ identifies with the subset defined by $z > 2$ and

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  -2 \cos(\theta/2) & \text{for a point with cone angle } \theta; 
  \end{cases}
  \]
Structures on $C_{1,1}$

- The Generalized Fricke space $\mathbb{H}'(C_{1,1})$ of $C_{1,1}$ identifies with the subset defined by $z > 2$ and

  $$Q_z(x, y) = x^2 + y^2 - zxy < 0.$$ 

- Trace function $z$ corresponding to two-sided interior curve $Z$.

- The boundary trace is:

  $$\delta := Q_z(x, y) + 2 = z^2 - k =$$
  
  \[\begin{cases}
  -2 \cosh(\ell/2) & \text{for a funnel with closed geodesic of length } \ell; \\
  -2 & \text{for a cusp}; \\
  -2 \cos(\theta/2) & \text{for a point with cone angle } \theta;
  \end{cases}\]

- Goldman – McShane – Stantchev – Ser Peow Tan
  
  *Automorphisms of two-generator free groups and spaces of isometric actions on the hyperbolic plane*, DG.1509.03790
The level set $\kappa^{-1}(k)$ for $k > 2$
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- Generalized Fricke space $\mathcal{F}'(C_{1,1})$ of $C_{1,1}$ projects to a linear sector in $\mathbb{R}^2$ invariant under

$$\text{Mod}(C_{1,1}) \cong \mathbb{Z}/2 \times (\mathbb{Z}/2 \ast \mathbb{Z}/2) \sim \langle Tw_Z \rangle \cong \mathbb{Z}.$$
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- Wandering domain under $\Gamma$ whose orbit is open and dense. What is the Hausdorff dimension of its complement?