Commensurators of thin normal subgroups

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Abstract. We give an affirmative answer to many cases of a question due to Shalom, which asks if the commensurator of a thin subgroup of a Lie group is discrete. In this paper, let $K < \Gamma < G$ be an infinite normal subgroup of an arithmetic lattice $\Gamma$ in a rank one Lie group $G$, such that the quotient $Q = \Gamma/K$ is infinite. We show that the commensurator of $K$ in $G$ is discrete whenever any one of the following holds:

1. $Q$ admits a linear representation with infinite image.
2. $\Gamma$ is not a non-uniform lattice in $G = \{SO(2,1), SO(3,1), SU(2,1)\}$ and $Q$ does not have Kazhdan’s property (T),
3. $Q$ acts semi-simply on a proper CAT(0) space and $\Gamma$ is not a non-uniform lattice in $G = \{SO(2,1), SO(3,1), SU(2,1)\}$. This is satisfied, in particular, if for such $\Gamma$, $Q$ admits a semi-simple representation into $SL_n(\mathbb{Q}(p))$.

Contents

1. Introduction 1
2. Generalities on discrete groups 5
3. Pseudo-actions 6
4. Hodge theory, patterns and commensurators 12
5. Abelian quotients and harmonic 1-forms 17
6. Non-abelian quotients and harmonic maps 23
7. Generalizations: Proof of items 2 and 3 of Theorem 1.4 28
Acknowledgments 31
References 32

1. Introduction

Let $G$ be a semi-simple $\mathbb{Q}$-algebraic group, and let $G(\mathbb{Z})$ denote its group of integer points. Roughly speaking, a subgroup $\Gamma$ of $G$ is called arithmetic if it is commensurable in a wide sense with $G(\mathbb{Z})$ [56]. That is, there is an element $g \in G$ such that the group $G(\mathbb{Z}) \cap \Gamma^g$ has finite index in both $G(\mathbb{Z})$ and $\Gamma^g$. In general, if $G$ is an algebraic group and $\Gamma < G$ is a subgroup, we write $\text{Comm}_{G}(\Gamma)$ for the commensurator of $\Gamma$ in $G$, i.e. the subgroup consisting of $g \in G$ such that
\( \Gamma \cap \Gamma^g \) has finite index in both \( \Gamma \) and \( \Gamma^g \). The Commensurability Criterion for Arithmeticity due to Margulis [49, 56] characterizes arithmetic subgroups of algebraic groups via their commensurators:

**Theorem 1.1** (Margulis). Let \( G \) be a connected semi-simple Lie group with no compact factors and let \( \Gamma \) be an irreducible lattice in \( G \). Then \( \Gamma \) is arithmetic if and only if \( \text{Comm}_G(\Gamma) \) is dense in \( G \).

In this article, we are primarily concerned with the discreteness properties of commensurators of subgroups of \( G \) which are not themselves lattices. We consider the class of thin groups instead, a class of groups which has received a large amount of attention in recent years [58]. Here, a subgroup \( \Gamma \subset G \) is thin if \( \Gamma \) is discrete and Zariski dense in \( G \), and if \( G/\Gamma \) has infinite volume with respect to Haar measure on \( G \). Thus, \( \Gamma \) fails to be a lattice in \( G \) by virtue of having infinite covolume in \( G \). Natural examples of thin groups arise from infinite index Zariski dense subgroups of lattices in \( G \).

In the present manuscript, we continue our previous investigations from [39] of the following question due to Shalom (see especially [63] where the problem has its genesis):

**Question 1.2.** [45] Let \( H \) be a thin subgroup of a semi-simple Lie group \( G \).

1. Is the commensurator \( \text{Comm}_G H \) of \( H \) in \( G \) discrete?
2. In particular, is the normalizer of \( H \) in \( G \) of finite index in \( \text{Comm}_G H \)?

Note that for a normal subgroup \( H \) of a lattice \( \Gamma \), the two sub-questions of Question 1.2 are equivalent. Positive answers to Question 1.2 are known for all finitely generated subgroups \( H \) of \( \text{PSL}_2(\mathbb{R}) \) and \( \text{PSL}_2(\mathbb{C}) \) [26, 45, 51], and for thin subgroups of semi-simple Lie groups with limit set a proper subset of the Furstenberg boundary [51]. Here, the limit set is a generalization of the limit set occurring in the theory of Kleinian groups, and is a minimal nonempty closed invariant subset of the Furstenberg boundary for a group acting on the corresponding symmetric space (see [5]).

We were thus prompted in [39] to address Question 1.2 when the ambient Lie group is the simplest possible, viz. \( \text{PSL}_2(\mathbb{R}) \), for thin groups whose limit sets consist of the entire Furstenberg boundary, i.e. \( S^1 = \partial \mathbb{H}^2 \). More generally, natural examples of thin groups with limit set all of the Furstenberg boundary come from normal subgroups of rank one lattices. It is this general problem that we address in this paper.

1.1. **Main results.** Since many rank one arithmetic lattices surject onto nonabelian free groups, every finitely generated group can be realized as a quotient of an arithmetic lattice. Observe in particular, that all finitely generated free groups arise as finite index subgroups of \( \Gamma(2) \), the level-two congruence subgroup of \( \text{PSL}_2(\mathbb{Z}) \). Thus, answering Question 1.2 for all normal subgroups of arithmetic lattices (or even \( \text{PSL}_2(\mathbb{Z}) \)) through their quotients implicitly implies a property of all finitely generated groups—a notoriously general class. This has led us to impose some natural algebraic/geometric conditions on \( Q \). We shall use the following somewhat non-standard definition in this paper for ease of exposition.

**Definition 1.3.** We shall say that a lattice \( \Gamma \) in a rank one Lie group \( G \) is a low-dimensional exception if

1. \( \Gamma \) is non-uniform, and
2. \( G \in \{ \text{SO}(2,1), \text{SO}(3,1), \text{SU}(2,1) \} \).

We will establish the following result, which handles normal subgroups with “nice” quotients.

**Theorem 1.4.** Let \( \Gamma < G \) be an arithmetic lattice in a rank one Lie group \( G \) and let \( K < \Gamma \) be an infinite normal subgroup. Write \( Q = \Gamma/K \) for the corresponding quotient group. Then the group \( \text{Comm}_G(K) \) is discrete in the following cases:
(1) The group \( Q \) admits an infinite image finite dimensional linear representation over \( \mathbb{C} \).

(2) The lattice \( \Gamma \) is not a low-dimensional exception (in the sense of Definition 1.3) and \( Q \) does not have Kazhdan’s property \((T)\).

(3) The lattice \( \Gamma \) is not a low-dimensional exception (in the sense of Definition 1.3), and \( Q \) admits a semi-simple action by isometries on a proper non-positively curved metric space. This hypothesis is verified, in particular, if for such \( \Gamma \) we have that the group \( Q \) admits a semi-simple representation into \( \text{SL}_n(\mathbb{Q}(p)) \).

Note that the hypotheses of the various cases of Theorem 1.4 are never satisfied for irreducible lattices in higher rank (due to Margulis’ normal subgroup theorem) and so Theorem 1.4 is vacuously true in these cases. In \([39]\) we answered Question 1.2 in the special case that \( H \) is the commutator subgroup of \( \Gamma \), where \( \Gamma < \text{PSL}_2(\mathbb{Z}) \) is a finite index normal subgroup of \( \text{PSL}_2(\mathbb{Z}) \) contained in a principal congruence subgroup \( \Gamma(k) \) for some \( k \geq 2 \). We vastly generalize this result, since if \( K = [\Gamma, \Gamma] \) has infinite index in \( \Gamma \) then \( K \) falls under the purview of item 1 of Theorem 1.4. In the case where \( [\Gamma, \Gamma] \) has infinite index in \( \Gamma \), we also conclude that \( \text{Comm}_G(K) \) is discrete whenever \( K \) is a normal subgroup corresponding to an “abelian” invariant, which is to say any term of the lower central series or derived series of \( \Gamma \) (cf. \([39]\)).

Item 2 of Theorem 1.4 applies especially in cases such as when \( Q \) is one the Thompson’s groups \( V, F \) or \( T \) and also when \( Q \) is an infinite amenable group. Such groups can be simple and thus may not have any finite linear representations \([35]\).

In light of item 3 of Theorem 1.4 (except for a low dimensional exceptions \( \Gamma \)), we have that the commensurator of \( K \) is discrete unless \( Q \) has property \((T)\) and admits no interesting action on a proper non-positively curved space. Thus \( Q \) needs to be rather exotic for the conclusion of Theorem 1.4 to fail.

1.2. Tools and Techniques. The main theorems and techniques of \([39]\) are the starting point of this paper. We mention at the outset a slightly non-standard convention that we shall use throughout the paper for ease of exposition.

**Convention 1.5.** We shall say that a property \( P \) holds for all \( N \gg 0 \) if \( N \in \mathbb{N} \) and \( P \) is satisfied for all \( n \in \mathbb{Z} \) (i.e. for all integer multiples of \( N \)).

**Pseudo-actions:** An important technical tool introduced in \([39]\) was that of a homology pseudo-action. Let \( S = X/\Gamma \), where \( \Gamma \) is a lattice in a semi-simple Lie group \( G \) as in Theorem 1.4, and where \( X \) is the associated symmetric space. An element \( g \in G \) is said to admit a homology pseudo-action on \( H_1(S, \mathbb{Q}) \) provided that for \([z] \in H_1(S, \mathbb{Q}) \), the homology class \( N g \cdot [z] = g \cdot [z^N] \) can be defined in \( H_1(S, \mathbb{Q}) \) for suitable \( N \neq 0 \) depending on \([z]\) and \( g \). We refine the homology pseudo-action in this paper to a pseudo-action on the quotient group \( Q = \Gamma/K \). For \( g \in G \), we say that the pseudo-action of \( g \) on \( Q \) is trivial if for all \( \gamma \in \Gamma \), there exists \( N \gg 0 \) (in the sense of Convention 1.5) such that
\[
\gamma^N \equiv (\gamma^N)g \pmod{K},
\]
as given in Definition 3.9. We direct the reader to Section 3 (particularly, Section 3.3) for details. A somewhat subtle point that arises in this paper is that while pseudo-actions may not even be well-defined, triviality of the pseudo-action is a well-defined notion. The following theorem, provides a rather general criterion for deciding non-commensurability (see Theorem 3.11):

**Theorem 1.6.** Let \( \Gamma < G \) and let \( K < G \) be normal. If \( g \in \text{Comm}_G(K) \cap \text{Comm}_G(\Gamma) \), then the pseudo-action of \( K^g = g^{-1}Kg \) on \( Q = \Gamma/K \) is trivial.
Theorem 1.6 allows us to define a *trivially pseudo-acting subgroup* \( \text{Comm}_{pa}(\Gamma, K) \) of \( G \) (Definition 3.10) and formulate a criterion (Proposition 3.15) that transfers the burden of the proof to showing that \( \text{Comm}_{pa}(\Gamma, K) \) is discrete.

**Harmonic Maps:** The other principal tools used in this paper come from harmonic maps via various incarnations of Hodge theory. These include classical (abelian) Hodge theory and its \( L^2 \)-analogue for non-compact manifolds, nonabelian Hodge theory following (principally) Corlette [14, 15], Labourie [44], and Simpson [64, 65], harmonic forms appearing in homological characterizations of Kazhdan’s property (T) [4, 55], and Hodge theory for general negatively curved spaces following Gromov [27], Gromov–Schoen [29], Korevaar–Schoen [40, 41], and Labourie [44]. Theorem 1.6 above allows us to roughly say that “coarse lines” in \( \Gamma \) (corresponding to infinite cyclic subgroups) are preserved modulo the normal subgroup \( K \). The harmonic map technology allows us to convert the coarse lines into actual maps from \( \mathbb{R} \), i.e. it allows us to “fill the holes” of coarse lines in a canonical fashion.

**Discrete patterns:** Harmonic maps are coupled with the notion of discrete patterns going back to Schwartz [59], and exploited in proving discreteness of commensurators in [45, 51]. Throughout the paper, we have tried to stress the basic example of arithmetic hyperbolic surfaces as well as the special case \( K = [\Gamma, \Gamma] \) separately, in order to explicate the underlying geometry. In the context of \( \text{PSL}_2(\mathbb{R}) \) and hyperbolic surfaces, Teichmüller–theoretic notions such as zeros and saddle connections of harmonic forms provide us the necessary discrete patterns that are preserved by the commensurator when the underlying surface has positive genus and the homology pseudo-action is trivial. Preservation of such discrete patterns finitely ensures that the commensurator is discrete. With the homology pseudo-action in place, the discussion for lattices in \( SO(n, 1) \) and \( SU(n, 1) \) splits into uniform and non-uniform cases. For uniform lattices, we use Hodge theory coupled with a Lie–theoretic idea that we learned from Venkataramana and Agol [66, 1]. For non-uniform lattices, we use \( L^2 \)-Hodge theory along with the fact that triviality of the homology pseudo-action guarantees the preservation of a discrete pattern given by horoballs. Discreteness of a pattern-preserving subgroup is an essential ingredient in the non-vanishing cuspidal cases: see Theorems 5.7 and 5.10.

**Relationship with existing literature:** The previous works [45, 51] on discreteness of commensurators derived discreteness by showing that the commensurator preserves a “discrete geometric sub-object” or “pattern” in the sense of Schwartz [60]. These may be regarded as a collection of geometrically defined subspaces of the domain symmetric space \( X \). There is a shift in focus in this paper, as we look at naturally defined geometric quotient or dual objects. The canonical nature of harmonic maps ensures that they are preserved by the commensurator.

Philosophically, the problem we address in this paper goes back to and derives inspiration from Shalom’s seminal work [62, 61, 63]. A particular point that only slowly became clear to us in hindsight is the following. One of the super-rigidity results of [61] takes lattices in locally compact topological groups as the domain and a rank one Lie group (\( SO(n, 1) \) or \( SU(n, 1) \)) as the target of a homomorphism. The aim there is to extend the homomorphism to the entire ambient topological group. Item (3) of Theorem 1.4 reverses the roles of the ambient Lie groups. The domain here is a lattice in \( SO(n, 1) \) or \( SU(n, 1) \). The target is a locally compact topological group—the isometry group of a proper non-positively curved space.

1.3. **Structure of the paper.** Section 2 contains an account of the general tools from the theory of lattices in Lie groups which we will need. Section 3 describes pseudo-actions (both homological and
general) in detail and places them on a sound footing. The relevance of pseudo-actions to studying commensurators of subgroups of lattices is spelled out there as well. Section 4 introduces the notion of a discrete invariant set as it arises from classical and $L^2$–Hodge theory. In the same section, the commensurator of a form is introduced and the construction of an invariant harmonic form from the dual pseudo-action arising from a trivial homological pseudo-action is carried out. Section 5 proves Theorem 1.4 in the case $K = [Γ, Γ]$, and more generally in the case where $b_1(Q) > 0$. Section 6 recalls the relevant background from nonabelian Hodge theory and allows us to complete the proof of Theorem 1.4 in the case that $Q$ admits an infinite image linear representation over $\mathbb{C}$. In Section 7, we complete the proofs of the other cases of Theorem 1.4. Generalizing the work in Section 5, we first prove Theorem 1.4 in the case that $Γ$ is not a low-dimensional exception and when $Q$ does not have Kazhdan’s Property (T). We also extend the work in Section 6 by proving Theorem 1.4 in the case that $Q$ admits a semi-simple representation into the isometry group of a proper CAT(0) space.

**Remarks on notation:** Throughout this paper, we will use the notation $K$ to denote a subgroup a discrete group. Oftentimes this will be a normal subgroup of an arithmetic lattice $Γ$. In particular, $K$ will generally not denote a maximal compact subgroup of the ambient Lie group $G$. We will use $N$ to denote a positive integer, as opposed to the more common notation of the unipotent subgroup in the Iwasawa decomposition of a semi-simple Lie group. We will use the exponentiation shorthand for conjugation in groups, so that $K^g = g^{-1}Kg$, where here $K$ and $g$ are contained in an ambient group. The group $G$ will denote an ambient Lie group, which will be assumed to be rank one and simple unless otherwise noted.

2. Generalities on discrete groups

In this section, we gather some general facts about Zariski dense discrete subgroups of semi-simple Lie groups which we will require in this article. The following Lemma generalizes the corresponding statement in [39] for $\text{PSL}_2(\mathbb{R})$.

**Lemma 2.1.** Let $Γ_0$ be a lattice in a rank one Lie group $G$. Let $Γ$ be a subgroup of $G$ containing $Γ_0$ such that there exists an $N > 0$ satisfying the property that for all $g ∈ Γ$, we have $g^N ∈ Γ_0$. Then $Γ$ is also discrete.

*Proof.* Since $Γ_0$ is a lattice, there exists $ε > 0$ such that any loxodromic (or equivalently, semi-simple) element of $Γ_0$ has translation length at least $ε$ under the canonical action of $G$ on its associated symmetric space. Since $G$ has rank one, it is simple. Hence, it follows that $Γ$ is either discrete or dense in $G$. We argue by contradiction. If $Γ$ is dense, then since the property of being loxodromic is an open condition and since translation lengths of loxodromic elements of $G$ coincide with $\mathbb{R}_{>0}$, there exists a loxodromic element $g ∈ Γ$ such that the translation length of $g$ is less than $ε/2N$. Hence $g^N$ is a loxodromic element with translation length at most $ε/2$. In particular, $g^N ∉ Γ_0$, which yields a contradiction. □

We also have the following general fact about normalizers of discrete groups. The proof is contained in [39], and so we omit it.

**Lemma 2.2.** Let $G$ be a simple Lie group and let $Γ < G$ be a discrete Zariski dense subgroup. Then the normalizer $N_G(Γ)$ is again discrete.

Let $G$ be a semi-simple Lie group and let $Γ < G$ be a subgroup. As usual, we write $\text{Comm}_G(Γ)$ to denote its commensurator in $G$. We shall need the following special case of a general theorem
of Borel [9, Theorem 2] (see [69, p. 123]). This will be the only real use of arithmeticity of the ambient lattice \( \Gamma \) in Theorem 1.4. Strictly speaking, the statement of Proposition 6.2.2 in [69] is for the integral points in an ambient group. The reader will note however that the only salient feature of the group of integral points which is used is its Zariski density. Thus, we obtain the following conclusion:

**Proposition 2.3.** Let \( \Gamma < G \) be an arithmetic lattice in a connected semi-simple algebraic \( \mathbb{Q} \)-group and let \( H < \Gamma \) be a Zariski dense subgroup. Then \( \text{Comm}_G(H) < \text{Comm}_G(\Gamma) \). Suppose furthermore that the center of \( G \) is trivial. Then \( \text{Comm}_G(\Gamma) \) coincides with the \( \mathbb{Q} \)-points of \( G \).

The hypothesis that \( G \) has trivial center in the second part of Proposition 2.3 is crucial. For instance, the commensurator of \( \text{SL}_2(\mathbb{Z}) \) properly contains \( \text{SL}_2(\mathbb{Q}) \). The reader will observe that throughout this paper, we will implicitly assume that \( K \) is a Zariski dense subgroup of an arithmetic lattice. In the statement of Theorem 1.4, we only assume that \( K \) is infinite and normal. This latter assumption implies that \( K \) is indeed Zariski dense:

**Proposition 2.4.** Let \( K < \Gamma \) be an infinite normal subgroup of an irreducible lattice in a semi-simple algebraic group \( G \). Then \( K \) is Zariski dense in \( G \).

**Proof.** Let \( \Lambda \) denote the limit set of \( K \). Since \( K \) is infinite, \( \Lambda \neq \emptyset \). Since \( K \) is normal, \( \Lambda \) coincides with the limit set of \( \Gamma [5] \). Since \( \Gamma \) is Zariski dense, so is \( K \). \( \square \)

The following technical fact will be used several times in this paper, and we extract it for modularity.

**Lemma 2.5.** Let \( K < G \) be a Zariski dense subgroup of a simple algebraic group \( G \), and let 

\[
K^G = \langle \{K^g \mid g \in \text{Comm}_G(K)\} \rangle
\]

be the subgroup of \( G \) generated by the conjugates of \( K \) by \( g \in \text{Comm}_G(K) \). If \( K^G \) is a discrete subgroup of \( G \), then \( \text{Comm}_G(K) \) is discrete.

**Proof.** We have immediately that \( K < \text{Comm}_G(K) \), since \( K \) normalizes itself. We therefore conclude that \( \text{Comm}_G(K) \) is either discrete or dense in \( G \). If \( \text{Comm}_G(K) \) is dense then there is a sequence \( g_i \to 1 \) of nontrivial group elements in \( \text{Comm}_G(K) \) converging to the identity. We write \( K_i = K^{g_i} \), and we observe that \( K_i < K^G \) for each \( i \). Choosing finitely many elements \( \{k_1, \ldots, k_m\} \subset K \) which generate a Zariski dense subgroup of \( G \), we have that if \( g_i \) is nontrivial then it cannot fix the entire collection \( \{k_1, \ldots, k_m\} \). However, as \( i \) tends to infinity, the conjugation action of \( g_i \) on \( \{k_1, \ldots, k_m\} \) tends to the identity. Thus, viewing \( G \) as a matrix group, we have that \( \{k_1^g, \ldots, k_m^g\} \) converges to \( \{k_1, \ldots, k_m\} \) in matrix norm. Since these elements lie in \( K_i < K^G \) which is discrete, we have that \( \{k_1^g, \ldots, k_m^g\} = \{k_1, \ldots, k_m\} \) element–wise for \( i \gg 0 \), which implies that \( g_i \) is the identity for \( i \gg 0 \) since \( G \) is simple and \( K \) is Zariski dense. This is a contradiction, and we conclude that \( \text{Comm}_G(K) \) is discrete. \( \square \)

The argument in Lemma 2.5 even shows that only the set \( \{K^g \mid g \in \text{Comm}_G(K)\} \) need be discrete in order to conclude the discreteness of \( \text{Comm}_G(K) \).

3. **Pseudo-actions**

In this section, we develop some ideas from homological algebra into one of the principal technical tools of this paper, well-adapted to the study of commensurations of normal subgroups of lattices in general and commutator subgroups in particular. The historical motivation comes from Chevalley–Weil theory which we recall first in the context of surfaces and free groups. Classical Chevalley–Weil
torsion–free so that no power of
The element
by Corollary 3.2.

homology class
imply that there are no inclusion relations between them. Let
lies in

Proof. Let
commutator subgroups
distinct finite index subgroups such that
Corollary 3.3.

notation
for the case of free groups. We recall a proof for the convenience of the reader. We shall use the
there is a homology class
Corollary 3.2.


Theorem 3.1
Chevalley–Weil Theory and homological pseudo-actions. Let
be an orientable surface (not necessarily closed) with nonabelian fundamental group. Let
 be a regular covering map with finite deck group
In [39], we used an explicit description of the
-module
in order to conclude that certain commensurations of
cannot commensurate
We recall this construction in a more general context.

3.1.1. Classical Chevalley–Weil Theory: surfaces. The following is the classical formulation of Chevalley–Weil Theory:

Theorem 3.1 (See [13, 23], cf. [38, 30]). Let
 be a finite regular cover of orientable surfaces of finite type with
nonabelian and with deck group

(1) If
 is closed of genus
 then
 is isomorphic to
 copies of the regular representation of
, together with two copies of the trivial representation of
.

(2) If
 is free of rank
 then
 is isomorphic to
 copies of the regular representation of
, together with one copy of the trivial representation of
.

The relevant consequence of Theorem 3.1 for our purposes is the following:

Corollary 3.2. Let
 be a covering space as in Theorem 3.1. Then for every
, there is a homology class
 such that

From Corollary 3.2, we can deduce the following consequence fairly easily, which appeared in [39] for the case of free groups. We recall a proof for the convenience of the reader. We shall use the notation
 to denote that the groups
 are abstractly isomorphic.

Corollary 3.3. Let
 be a finitely generated surface group or free group, and let
 be distinct finite index subgroups such that
 is normal in
, and such that
 . Then the commutator subgroups
 and
 are not commensurable.

Proof. We will exhibit an element
 such that
 but such that no power of
 lies in
. Let
. Such an element
 exists because the hypotheses on
 and
 imply that there are no inclusion relations between them. Let
 be an element representing a homology class
 which is not fixed by
. The existence of such a
 is guaranteed by Corollary 3.2.

We can choose
, such that
 and
. Thus, we have that
, but at the level of the homology of
, we have that

The element
 is the desired element, as now follows from the fact that the abelianization of
 is torsion–free so that no power of
 represents a trivial homology class.

□
3.1.2. The groupoid of paths and pseudo-actions: CW complexes. An unsatisfying hypothesis in Corollary 3.3 is the assumption that $K_1$ is normal in $G$. It turns out that for our purposes at least, we can eschew this hypothesis. This is roughly because we are uninterested in a precise description of the action of an ambient deck group on the homology of a cover; rather we just wish to identify a homology class which is not fixed. Thus, we consider a more general setup where $S$ is a general CW complex and $p: S' \to S$ is a finite cover which is connected but not necessarily regular.

Since $\pi_1(S')$ is not normal in $\pi_1(S)$, we cannot make sense of a conjugation action of $\pi_1(S)$ on $\pi_1(S')$, though we can make sense of the action of $\pi_1(S)$ on the groupoid of paths in $\pi_1(S')$. The groupoid of paths we consider here is closely related to but somewhat different from the fundamental groupoid. Fix a basepoint $\ast \in S$ and identify elements of $\pi_1(S)$ with oriented loops in $S$ based at $\ast$. Loops in $S$ based at $\ast$ lift to oriented paths based at points in $p^{-1}(\ast)$. Concatenation of oriented paths based at $p^{-1}(\ast)$ is evidently a groupoid, and there is an obvious action of $\pi_1(S)$ which changes a lift of a path by permuting basepoints: this action needs to be suitably interpreted in the context of groupoids as follows. Specifically, let $c$ be an oriented loop at $\ast$ and let $\gamma \in \pi_1(S)$. If $\tilde{c} \in p^{-1}(\ast)$, then there is a unique lift $\tilde{\gamma}(1)$, where $\tilde{\gamma}$ is the lift of $\gamma$, viewed as an oriented loop in $S$, starting at $\tilde{\ast}$.

If $c$ is an oriented loop based at $\ast$ then one can consider the homology class of $c$ in $H_1(S, \mathbb{Q})$. The homology class of a lift $\tilde{c}$ is generally not well defined, though there is always a positive $N$ such that $\tilde{c}/N$ has a well-defined homology class, once a particular lift of $\ast$ has been chosen. Similarly, one may not be able to define the homology class of $\gamma \cdot \tilde{c}/N$, though by replacing $N$ by some multiple if necessary, we can arrange for both $\tilde{c}/N$ and $\gamma \cdot \tilde{c}/N$ to have well-defined homology classes.

Thus, we have that $\pi_1(S)$ admits a pseudo-action on $H_1(S', \mathbb{Q})$ in the sense that if $g \in \pi_1(S)$ and $[z] \in H_1(S', \mathbb{Q})$, the homology class $N g \cdot [z] = g \cdot [z]/N$ can be defined as a homology class of $S'$ for $N$ sufficiently large depending on $[z]$ and $g$. We say that this pseudo-action is nontrivial if there is a $q \in \pi_1(S)$ and $[z] \in H_1(S', \mathbb{Q})$ such that $g \cdot [z]/N \neq [z]/N$ in $H_1(S', \mathbb{Q})$, and otherwise we say that the pseudo-action is trivial. We can thus generalize Corollary 3.3 to the case of non–regular covers in the case of groups under quite general hypotheses, and thus obtain the following non-commensurability criterion:

**Theorem 3.4.** Let $G$ be a group and let $K_1, K_2 < G$ be commensurable subgroups of finite index. If the pseudo-action of $K_2$ on $H_1(K_1, \mathbb{Q})$ is nontrivial then the commutator subgroups $K_1'$ and $K_2'$ are not commensurable.

**Proof.** The proof is identical to that of Corollary 3.3, setting $G = \pi_1(S)$ in the preceding discussion. That is, the hypotheses of the theorem imply the existence of $g \in K_2$ and $z \in K_1$ such that $g \cdot [z]/N \neq [z]/N$ in $H_1(K_1, \mathbb{Q})$ for a suitably chosen $K_2' > 0$ (cf. Convention 1.5). Then the element $x = [g, z]/N$ witnesses that $K_1'$ and $K_2'$ are not commensurable. \hfill $\square$

Using Theorem 3.4, we generalize Corollary 3.3 as follows:

**Corollary 3.5.** Let $G < \text{PSL}_2(\mathbb{R})$ be a lattice, and let $K_1, K_2 < G$ be distinct torsion–free subgroups of finite index such that $K_1 \cong K_2$. Then the groups $K_1'$ and $K_2'$ are not commensurable.

**Proof.** As before, we have that there is an element $g \in K_2 \setminus K_1$, since Euler characteristic considerations imply that there are no inclusion relations between $K_1$ and $K_2$. By Theorem 3.4, it suffices to find an element $z \in K_1$ such that the homology classes $[z]$ and $g \cdot [z]$ in $H_1(K_1, \mathbb{Q})$ are defined, and such that $[z] \neq g \cdot [z]$. With this setup, the group $K_2'$ becomes irrelevant, and we just consider $K_1 = K < G$. 

If $K$ is normal in $G$ and $g \in G \setminus K$ then $g$ acts on $K$ by conjugation. The element $g$ can thus be viewed as an isometry of a hyperbolic surface $S$, which is the manifold cover of $\mathbb{H}^2/G$ classified by $K$. The isometry $g$ acts nontrivially on $H_1(S, \mathbb{Q})$, as is a standard result about finite order mapping classes [20]. In particular, there is a homology class $z \in H_1(S, \mathbb{Q})$ which is not fixed by $g$.

Thus, we may assume that $K$ is not normal in $G$. We choose a finite generating set for $K$ which we write as $\{z_1, \ldots, z_n\}$ such that each $z_i$ represents a primitive integral homology class of $K$. We may choose these loops to each be represented by simple closed curves on $S$. Fix such closed loop representatives of $z_i$ in $S$. If in addition $g \cdot z_i$ is again a closed loop for each $i$ (with this action interpreted as in the groupoid of paths), then it follows that $K$ is normal. Thus, we may assume that there is an $i$ for which $g \cdot z_i$ is not a closed loop. We let $N > 1$ be the smallest positive integer so that $g \cdot z_i^N$ is a closed loop. Since $z_i$ was simple to begin, we have that $g \cdot z_i^N$ is again simple.

The integral homology class of an oriented simple loop on $S$ is either zero or it is primitive. It follows that $0 \neq \langle z_i^N \rangle$ is not a primitive homology class, and $\langle g \cdot z_i^N \rangle$ is either zero or a primitive homology class. In either case, we have that $\langle z_i^N \rangle \neq \langle g \cdot z_i^N \rangle$ in $H_1(K, \mathbb{Q})$. The corollary now follows.

We will require a further manifestation of pseudo-actions which is well-suited for commensurable lattices which do not lie in a common lattice. For this, we consider $\Gamma_1, \Gamma_2$ to be subgroups of a semi-simple Lie group $G$ which are commensurable with each other.

We will construct a pseudo-action of $\Gamma_2$ on $H_1(\Gamma_1, \mathbb{Q})$ by defining a suitable analogue of the groupoid of paths. In this case, the pseudo-action is defined by passing to $X$, which is the symmetric space associated to $G$. The space $X$ is then the orbifold universal cover of the spaces $X/\Gamma_1$ and $X/\Gamma_2$.

Let $\ast$ be a fixed basepoint of $X/\Gamma_1$ and let $c \subset X/\Gamma_1$ be an oriented loop based at $\ast$. If $\gamma \in \Gamma_2$ then we define $\gamma \cdot c$ by choosing a lift $\tilde{c}$ of $c$ to $X$, applying the isometry $\gamma \in G$, and looking at the quotient under the action of $\Gamma_1$. Thus, $\gamma \cdot c$ will be a path in $X/\Gamma_1$ which will generally not be closed.

Let $z$ be a closed oriented loop in $X/\Gamma_1$, so that we may consider the homology class $[z]$, and let $\gamma \in \Gamma_2$. We may choose an $N > 0$ such that $\gamma \cdot [z^N]$ also represents a homology class of $X/\Gamma_1$. This is the pseudo-action of $\Gamma_2$ on the homology of $\Gamma_1$. We say that the pseudo-action of $\gamma$ is trivial if $\gamma \cdot [z^N] = [z^N]$ for all $z$ and $N \geq 0$ (cf. Convention 1.5) such that the two sides of this equation are homology classes of $X/\Gamma_1$.

The following non-commensurability criterion follows immediately by an argument identical to that of Theorem 3.4.

**Corollary 3.6.** Let $\Gamma_1, \Gamma_2 \lhd G$ be commensurable subgroups, and suppose that the pseudo-action of $\Gamma_2$ on $H_1(\Gamma_1, \mathbb{Q})$ is nontrivial. Then the corresponding commutator subgroups $\Gamma'_1$ and $\Gamma'_2$ are not commensurable.

We state explicitly the following particular case of Corollary 3.6:

**Corollary 3.7.** Let $\Gamma_1, \Gamma_2 \lhd \text{PSL}_2(\mathbb{R})$ be commensurable torsion–free lattices. Suppose that for some $\gamma \in \Gamma_2$, the pseudo-action of $\gamma$ on $H_1(\Gamma_1, \mathbb{Q})$ is nontrivial. Then the groups $\Gamma'_1$ and $\Gamma'_2$ are not commensurable.

### 3.2. Cohomological pseudo-actions

One complicating feature of homology pseudo-actions is that they are not well-defined actions. If $\Gamma_1$ and $\Gamma_2$ are contained in an ambient group $G$, then the conjugation action of $\Gamma_2$ on $\Gamma_1$ is well-defined since it occurs in $G$. However, passing to homology classes introduces some difficulties. For instance, let $z \in \Gamma_1$, and suppose that $[gz^Ng^{-1}] \neq [z^N] \in \mathbb{H}^2$.
exists an trivial homological pseudo-action is well-defined when we pass between to produce an invariant harmonic form. consider the form homology of cohomological pseudo-action in a more canonical way by identifying action of for each sufficiently high multiple. That is, choose sup g such that the pseudo-action of Suppose therefore that the pseudo-action of $\Gamma_2$ on $H_1(\Gamma_1, \mathbb{Q})$ is trivial. In this case, the pseudo-action is well defined since
\[ g \cdot [z^N] = [gz^N g^{-1}] = [z^N] \in H_1(\Gamma_1, \mathbb{Q}), \]
and thus if $w$ is homologous to $z$ and if $[gz^N g^{-1}]$ and $[gw^N g^{-1}]$ are both elements of $H_1(\Gamma_1, \mathbb{Q})$, then
\[ [z^N] = [gz^N g^{-1}] = [gw^N g^{-1}] = [w^N] \in H_1(\Gamma_1, \mathbb{Q}). \]
Thus, we see that the homological pseudo-action is well-defined provided it is trivial.

We retain the setup of commensurable lattices $\Gamma_1, \Gamma_2 < G$ for $G$ a semi-simple Lie group. We suppose that the pseudo-action of $\Gamma_2$ on $H_1(\Gamma_1, \mathbb{Q})$ is trivial. The triviality of the pseudo-action of $\Gamma_2$ on $H_1(\Gamma_1, \mathbb{Q})$ now dualizes to a trivial pseudo-action of $\Gamma_2$ on $H^1(\Gamma_1, \mathbb{Q})$ as follows. Let $g \in \Gamma_2$ be fixed and let $\{z_1, \ldots, z_n\} \subset \Gamma_1$ be elements which represent generators of $H_1(\Gamma_1, \mathbb{Q})$. The pseudo-action of $g$ on $H_1(\Gamma_1, \mathbb{Q})$ is additive in the sense that if $g \cdot [z_i^N]$ and $g \cdot [z_j^N]$ are well-defined homology classes, then $g \cdot [z_i^N z_j^N] = g \cdot [z_i^N] + g \cdot [z_j^N]$.

If $g \in \Gamma_2$ then $g$ admits a dual pseudo-action on $H^1(\Gamma_1, \mathbb{Q})$ by replacing $\omega \in H^1(\Gamma_1, \mathbb{Q})$ by a sufficiently high multiple. That is, choose $N$ such that $g \cdot [z_i^N]$ represents a homology class of $\Gamma_1$ for each $i$. Then,
\[ (g \cdot (N \omega))(\{z_i\}) = g \cdot \omega(N [z_i]) = g \cdot \omega([z_i^N]) = \omega(g \cdot [z_i^N]). \]
Here, we take this sequence of equalities to be the definition of $g \cdot (N \omega)$, since a priori there is no action of $g$. If $G$ is a semi-simple Lie group with associated symmetric space $X$, we can view the cohomological pseudo-action in a more canonical way by identifying $H_1(\Gamma_1, Q)$ with the rational homology of $X/\Gamma_1$. Then, we can pull back $\omega$ to a form $p^* \omega$ on $X$, and if $g \in G$ then we can then consider the form $g^* p^* \omega$. We will spell out this interpretation of the cohomological pseudo-action in Section 4 below, specifically Subsection 4.2, where we use a trivial cohomological pseudo-action to produce an invariant harmonic form.

Returning to the definition, since the pseudo-action of $\Gamma_2$ on $H_1(\Gamma_1, \mathbb{Q})$ is trivial, then for each $g \in \Gamma_2$ there is an $N$ for which $g \cdot (N \omega) = N \omega \in H^1(\Gamma_1, \mathbb{Q})$. Note that we are using the fact that the trivial homological pseudo-action is well-defined when we pass between $N[z_i]$ and $[z_i^N]$. Indeed, we are using the fact that $g \cdot [z_i^N]$ is independent of the choice of $z_i$, up to possibly replacing $N$ by a higher exponent (cf. Convention 1.5). Observe that since $\Gamma_1 \cap \Gamma_2$ has finite index in both $\Gamma_1$ and $\Gamma_2$, there is a universal $N$ such that $g \cdot [z_i^N]$ is a homology class in $H_1(\Gamma_1, \mathbb{Q})$, independent of $z$.

Thus, we say that the dual pseudo-action of $\Gamma_2$ on $H^1(\Gamma_1, \mathbb{Q})$ is trivial if for all $g \in \Gamma_2$ there exists an $N \gg 0$ such that for all $\omega \in H^1(\Gamma_1, \mathbb{Q})$, we have
\[ g \cdot \omega([z^N]) = \omega(g \cdot [z^N]) = \omega([z^N]). \]

We thus conclude:
Corollary 3.8. Let $\Gamma_1, \Gamma_2$ be commensurable lattices in a semi-simple Lie group $G$, and suppose that the pseudo-action of $\Gamma_2$ on $H_1(\Gamma_1, \mathbb{Q})$ is trivial. Then the dual pseudo-action of $\Gamma_2$ on $H^1(\Gamma_1, \mathbb{Q})$ is also trivial.

The additivity of the pseudo-action allows for triviality to be checked on basis elements only.

3.3. General pseudo-actions and a discreteness criterion. We now depart from the homological world and develop a robust criterion for commensuration which applies to general quotient groups and not just abelian ones. Let $\Gamma < G$, let $K < \Gamma$ be a normal subgroup, and let $g \in G$ commensurate $\Gamma$. We write $Q = \Gamma/K$ for the quotient group. Conjugating by $g \in G$, we obtain groups $K^g < \Gamma^g$ and a corresponding quotient $Q^g = \Gamma^g/K^g$. We now develop a pseudo-action criterion for commensurations of $K$.

Definition 3.9. Let $\Gamma < G$, $K$ be a normal subgroup of $\Gamma$ and $Q = \Gamma/K$. Let 
\[ g \in \Comm_G \Gamma \cap \Comm_G K. \]
We say that the pseudo-action of $g$ on $Q$ is trivial if for all $\gamma \in \Gamma$, we have that 
\[ \gamma^N \equiv (\gamma^N)^g \pmod{K} \]
for $N > 0$. That is, there exists $N > 0$ such that $x_m = [\gamma^N, g] \in K$ for all $m \in \mathbb{Z}$.

Definition 3.10. Let $\Gamma < G$; let $K$ be a normal subgroup of $\Gamma$ and $Q = \Gamma/K$. The trivially pseudo-acting submonoid $\Comm^M_{\text{pa}}(\Gamma, K)$ of $G$ is defined to be the submonoid consisting of all $g \in G$ such that the pseudo-action of $g$ on $Q$ is both defined and trivial. It is straightforward to check that $\Comm^M_{\text{pa}}(\Gamma, K)$ contains the identity and is closed under multiplication of elements. However, it is not clear that inversion of elements is possible within $\Comm^M_{\text{pa}}(\Gamma, K)$.

Theorem 3.11. Let $\Gamma < G$. Let $K$ be a normal subgroup of $\Gamma$ and let $Q = \Gamma/K$. Let 
\[ g \in \Comm_G \Gamma \cap \Comm_G K. \]
Then the pseudo-action of $K^g$ on $Q$ is trivial, i.e. $K^g \subset \Comm^M_{\text{pa}}(\Gamma, K)$.

Proof. Let $z \in K^g$ and let $\gamma \in \Gamma$ be arbitrary fixed elements. For $N > 0$ we have that $\gamma^N \in \Gamma \cap \Gamma^g$ and $(\gamma^N)^z \in \Gamma$. Let $a = (\gamma^N)^z$ and $b = \gamma^N$. We have that $a^m, b^m \in \Gamma$ for all $m \in \mathbb{Z}$.

Since $z \in K^g$ and since $K^g$ is normal in $\Gamma^g$, we have that 
\[ a \equiv b \pmod{K^g}. \]
Hence, 
\[ a^m \equiv b^m \pmod{K^g} \]
for all $m \in \mathbb{Z}$. Thus, the commutators 
\[ x_m := [\gamma^N, z] = a^m b^{-m} \]
have the property that $x_m \in K^g$ for all $m \in \mathbb{Z}$. It is also clear that $x_m \in \Gamma$ for all $m \in \mathbb{Z}$.

Since $K$ and $K^g$ are commensurable, the collection of elements 
\[ \{x_m = a^m b^{-m}\}_{m \in \mathbb{Z}} \]
has the property that for some $s \neq t$, the elements $x_s = a^s b^{-s}$ and $x_t = a^t b^{-t}$ lie in the same right coset of $\Gamma \cap K^g$ in $K^g$, as follows immediately from the pigeonhole principle.

It follows that there exists an element $k \in K$ such that 
\[ k a^s b^{-s} = a^t b^{-t}. \]
Therefore, we see that
\[ a^{−t}ka^s = b^{s−t}, \]
which furnishes an element \( k' \in K \) such that \( k'\alpha^{s−t} = b^{s−t} \).

Thus, there exists \( M = s−t \neq 0 \) such that \( a^M \equiv b^M \pmod{K} \). In particular, the pseudo-action of \( z \) on \( Q \) is trivial. Since \( z \) was an arbitrary element of \( K^g \), the pseudo-action of \( K^g \) on \( Q \) is trivial.

The following immediate corollary says that for \( Q \) a torsion–free quotient of \( \Gamma \) by a normal subgroup \( K \), we need only to check the failure of Theorem 3.11 for a single element of \( \Gamma \).

**Corollary 3.12.** Let \( \Gamma < G \) and let \( K \) be a normal subgroup of \( \Gamma \) such that \( Q = \Gamma/K \) is torsion–free. Let \( g \in \text{Comm}_G \Gamma \) and \( z \in K^g \). For \( \gamma \in \Gamma \), let \( N \) be such that \( \gamma^N \in \Gamma \cap \Gamma^g \) and \( (\gamma^N)_z \in \Gamma \). If \( x = [\gamma^N, z] \notin K \), then \( K \) and \( K^g \) are not commensurable.

When \( b_1(Q) > 0 \) then the above proof furnishes the following commensurability criterion.

**Theorem 3.13.** Suppose that \( K \) and \( K^g \) are commensurable. Then the pseudo-action of \( K^g \) on \( H_1(Q, \mathbb{Q}) \) is trivial. Dually, the pseudo-action of \( K^g \) on \( H^1(Q, \mathbb{Q}) \) is trivial.

**Definition 3.14.** Theorem 3.11 allows us to define the notion of a group \( \text{Comm}_{pa}(\Gamma, K) \). Namely, we define a \emph{trivially pseudo-acting subgroup} \( \text{Comm}_{pa}(\Gamma, K) \) of \( G \) to be a maximal subgroup of \( \text{Comm}_{pa}^M(\Gamma, K) \) which contains \( K^G = \langle \{K^g \mid g \in \text{Comm}_G(\Gamma) \} \rangle \).

It is a straightforward application of Zorn’s Lemma to prove that a maximal subgroup \( \text{Comm}_{pa}(\Gamma, K) \) of \( \text{Comm}_{pa}^M(\Gamma, K) \subset G \) exists. We now observe the following discreteness criterion for \( \text{Comm}_G(\Gamma) \):

**Proposition 3.15.** Let \( G \) be a rank one Lie group, \( \Gamma \) an arithmetic lattice and \( K < \Gamma \) an infinite normal subgroup. Then \( \text{Comm}_G(\Gamma) \) is discrete provided \( \text{Comm}_{pa}(\Gamma, K) \) is discrete.

**Proof.** Let \( g \in \text{Comm}_G(\Gamma) \). It follows from Proposition 2.3 that \( g \in \text{Comm}_G(\Gamma) \cap \text{Comm}_G(\Gamma) \). Hence by Theorem 3.11, we have an inclusion \( K^g \subset \text{Comm}_{pa}(\Gamma, K) \), as feeds into Definition 3.14. Hence, the \( \text{Comm}_G(\Gamma) \)–orbit of \( K \) lies inside the discrete subgroup of \( \text{Comm}_{pa}(\Gamma, K) \). In particular, the group \( K^G \) is discrete. The Proposition follows from Lemma 2.5.

4. Hodge theory, patterns and commensurators

The goal of this section is to translate between triviality of the homological pseudo-action and the existence of commensuration-invariant geometric objects. The geometric objects we consider are harmonic 1-forms and discrete patterns. These will provide us with the essential tools to conclude discreteness of commensurators.

4.1. Hodge theory and discrete invariant sets

In this subsection we recall some essential tools on harmonic forms and discrete patterns from existing literature.

4.1.1. The Hodge theorem and \( L^2 \)–cohomology: We recall the necessary tools from Hodge theory and \( L^2 \)–cohomology that we shall need. Let \( M \) be a (not necessarily compact) Riemannian manifold. We fix notation: \( \Omega^k \) will denote the space of smooth \( k \)–forms, \( d \) will denote the differential on forms, \( * \) will denote the Hodge star operator, \( d^* \) will denote the adjoint of \( d \), and \( \Delta = dd^* + d^*d \) will denote the Laplacian on forms. A form \( \omega \in \Omega^k \) is a \emph{harmonic \( k \)–form} for the given metric on \( M \) if \( \Delta \omega = 0 \). Harmonic forms are closed and co-closed.

**Theorem 4.1.** [67, Ch. 6] Let \( M \) be a compact Riemannian manifold. Then for all \( k \) and every \( \omega \in H^k(M, \mathbb{R}) \), there exists a unique harmonic form \( \omega_{\text{harm}} \in [\omega] \).
We shall need a version of Theorem 4.1 for non-compact complete manifolds $M$. The appropriate cohomology theory used is $L^2$-cohomology. Let $\Omega^k_0$ denote the space of smooth square-integrable $k$-forms. The reduced $L^2$-cohomology groups are given by

$$H^i_{\text{red}}(M) = \ker(d)/\overline{\text{Im}(d)},$$

where $\overline{\text{Im}(d)}$ denotes the closure of the image of $d$. We refer the reader to [11] for more details. We shall need only the following special case (see [11, Lemma 1.5] due to Gaffney, or [12] for instance):

**Theorem 4.2.** Let $M$ be a complete negatively curved manifold of finite volume modeled on $\mathbb{H}^n$ or $\mathbb{C}H^n$. Then for every real cohomology class $[\omega] \in H^1_{\text{red}}(M, \mathbb{R})$, there exists a unique $L^2$ harmonic form $\omega_{\text{harm}} \in [\omega]$.

**4.1.2. Discrete Patterns:** Let $G$ be a rank one connected semi-simple Lie group and let $X$ be the associated symmetric space. The space $X$ is, in a natural way, a Riemannian manifold endowed with a left-invariant metric [31]. Following [60, 59, 52, 8] we define the following (see [52, Definition 1.6] in particular):

**Definition 4.3.** Let $\Gamma < G$ be a lattice and $S = X/\Gamma$. A $\Gamma$–discrete pattern of points on $X$ is a non-empty $\Gamma$–invariant set $S \subset X$ such that $S/\Gamma$ is finite. A $\Gamma$–discrete pattern of geodesics on $X$ is a non-empty $\Gamma$–invariant collection $S \subset X$ of bi-infinite geodesics such that $S/\Gamma$ is a finite collection of geodesics on $S$ whose union is a closed subset of $S$.

**Definition 4.4.** Let $\Gamma < G$ be a lattice. A subgroup $H$ of $G$ is said to preserve a $\Gamma$–discrete pattern $S$ (of points or geodesics) if $h(S) \subset S$ for all $h \in H$.

Propositions 3.5 and 3.7 of [52] show that a subgroup $H$ of $G$ preserving a $\Gamma$–discrete pattern $S$ is closed and totally disconnected. Since any such subgroup of $G$ is necessarily discrete, we have the following:

**Lemma 4.5.** [52, Propositions 3.5 and 3.7] Let $\Gamma < G$ be a lattice and $S$ a $\Gamma$–discrete pattern (of points or geodesics). Then the subgroup $H$ of $G$ preserving $S$ is discrete, and $[H : \Gamma] < \infty$.

**Definition 4.6.** Let $G$ and $X$ be as before, let $\Gamma < G$ be a non-uniform lattice, and let $S = X/\Gamma$. A $\Gamma$– discrete pattern of horoballs in $X$ is a non-empty $\Gamma$–invariant collection $S \subset X$ of closed horoballs such that $S/\Gamma$ is a disjoint union of neighborhoods of cusps.

Propositions 5.3 and 5.4 of [52] (see also [50, Theorem 3.11]) prove that the subgroup $H$ of $G$ preserving a $\Gamma$–discrete pattern of horoballs is closed and totally disconnected. Hence it is discrete. Since $\Gamma < H$, this forces $[H : \Gamma] < \infty$. We explicitly state this below for future reference.

**Lemma 4.7.** [52, Propositions 5.3 and 5.4] Let $\Gamma < G$ be a non-uniform lattice in a rank one Lie group and $S = X/\Gamma$, where $X$ is the associated symmetric space. Let $S^\circ$ be a $\Gamma$–discrete pattern of horoballs. Then the subgroup $H$ of $G$ preserving $S$ is discrete, and $[H : \Gamma] < \infty$.

**4.2. Actions on forms.** We shall need to set up some notation for the purpose of this section. The arguments in this subsection are general and work for arbitrary semi-simple Lie groups $G$. Let $X$ denote the associated symmetric space of noncompact type, and let $\Gamma$ be an arithmetic lattice in $G$ and $g \in \text{Comm}_G(\Gamma)$. We denote $S = X/\Gamma$ and $S^\circ = X/\Gamma^\circ$. Since $g \in \text{Comm}_G(\Gamma)$, $\Gamma \cap \Gamma^\circ$ is of finite index in both $\Gamma$ and $\Gamma^\circ$. Let $W = X/(\Gamma \cap \Gamma^\circ)$ denote the corresponding common cover of $S$ and of $S^\circ$. We shall refer to $S$ and $S^\circ$ as conjugate manifolds and $W$ as their minimal common cover. Let $p : X \to S$ denote the universal covering map. For $\omega$ a harmonic (or $L^2$–harmonic)
1-form on $S$, the form $p^*\omega$ is a harmonic 1-form on $X$. Since $g$ acts isometrically on $X$, the form $g^*p^*\omega$ is a harmonic 1-form on $X$ which is invariant under $\Gamma^g$ and hence descends to $S^g$. The resulting harmonic (or $L^2$-harmonic) 1-form on the quotient manifold $S^g$ is denoted as $\omega^g$. Let $q : W \to S$ and $q^g : W \to S^g$ denote the natural covering maps. Denote $q^*\omega$ by $\omega_W$ and $(q^g)^*\omega^g$ by $\omega_W^g$.

We shall also need to set up notation for $g$–conjugates of cycles and loops as base-points will play an important role in what follows. Let $o \in W$ be a base-point. By choosing a lift $\tilde{o} \in X$ and by joining $\tilde{o}$ to $g.\tilde{o}$ by a geodesic segment in $X$ and projecting back to $W$, we obtain a natural geodesic segment $[o, g.o]$ in $W$, where $g.o$ denotes the image of $g.\tilde{o}$ under the covering projection. Thus, $g.o$ may be regarded as a new base-point.

Now suppose that $\alpha$ is a loop in $W$ representing an element $h \in \pi_1(W)$ such that $h^g$ also belongs to $\pi_1(W)$, where $\pi_1(W)$ is identified with $\Gamma \cap \Gamma^g$. Lifting $\alpha$ to a path $\tilde{\alpha}$ in $X$, translating by $g$ and quotienting $X$ by $\Gamma \cap \Gamma^g$ we obtain a new loop denoted $g.\alpha$ on $W$ based at $g.o$. Here, we use notation that is similar to the case of a genuine $g$–action on $W$, though the action is well-defined only on the universal cover $X$.

The concatenation $[o, g.o] * g.\alpha * [o, g.o]$ gives a loop based at $o$, where $[o, g.o]$ denotes $[o, g.o]$ parametrized in the opposite direction from $g.o$ to $o$. We denote this loop as $\alpha^g$:

$$\alpha^g = [o, g.o] * g.\alpha * [o, g.o].$$

Finally, for $\sigma$ any closed loop on $W$, based at $o$ say, the $n$–th power of the loop $\sigma$ will be the loop which traverses the loop $\sigma$ a total of $n$ times. The result will be denoted by $\sigma^n$. The next Lemma gives a Hodge-theoretic version of Corollary 3.8. Much of the proof of Lemma 4.8 is formal; however, since the pseudo-action is not necessarily an action we provide details.

**Lemma 4.8.** For

$$\{\Gamma, S, g, S^g, W, \omega_W, \omega^g_W\}$$

as above, if the homology pseudo-action of $g$ on $H_1(S, \mathbb{Q})$ is trivial, then $\omega_W = \omega^g_W$.

**Proof.** Recall from Section 3 that if the homology pseudo-action of $g$ on $H_1(S, \mathbb{Q})$ is trivial, then there exists $m > 0$ such that $g_* : H_1(S, m\mathbb{Z}) \to H_1(S, m\mathbb{Z})$ is the identity.

We carry on the notation from the discussion before the Lemma. Let $\sigma$ be any closed loop on $W$ based at $o$. Choose $n > 0$ such that $\sigma^n$ and $(\sigma^n)^g$ are well-defined loops based at $o$. Note that this is possible if and only if both

$$h, h^g \in \Gamma \cap \Gamma^g,$$

where $h$ denotes the element of $\pi_1(W, o)$ represented by $\sigma^n$. Also, note that $(\sigma^n)^g$ represents $h^g$.

Triviality of the homology pseudo-action of $g$ on $H_1(S, \mathbb{Q})$ implies the existence of an $N > 0$ such that $\sigma^N$ and $(\sigma^N)^g$ represent homologous elements in $H_1(S, \mathbb{Q})$. Hence,

$$\int_{q((\sigma^N)^g)} \omega = \int_{q((\sigma^N)^g)} \omega,$$

where $q : W \to S$ is the covering projection.

Next, we have that

$$\int_{\sigma^N} \omega_W = \int_{q((\sigma^N)^g)} \omega,$$

and

$$\int_{(\sigma^N)^g} \omega_W^g = \int_{q((\sigma^N)^g)} \omega,$$

by the definition of $\omega_W$, and so we obtain
However, we have that
\[ \int_{(\sigma N)^g} \omega_W = \int_{g(\sigma N)} \omega_W, \]
since the integrals along \([o,g,o]\) and \([o,g,o]\) cancel each other.

Finally, we observe (by lifting to \(X\) and acting by \(g\) on \(X\) for instance) that
\[ \int_{g(\sigma N)} \omega_W = \int_{p^N} \omega_W^g. \]

Putting all these equalities together, we finally have,
\[ \int_{\sigma N} \omega_W = \int_{\sigma N} \omega_W^g. \]

Since
\[ \int_{\sigma} \omega_W = N \int_{\sigma} \omega_W, \]
we conclude that
\[ (1) \quad \int_{\sigma} \omega_W = \int_{\sigma} \omega_W^g \]
for any closed loop \(\sigma\) in \(W\) based at \(o\). Since the forms \(\omega_W\) and \(\omega_W^g\) are harmonic, they represent well-defined elements of \(H^1(W, \mathbb{R})\). By Equation 1, they represent the same element of \(H^1(W, \mathbb{R})\).

So far we have not invoked the harmonicity of \(\omega_W\) and \(\omega_W^g\). We do so now. It follows from Theorem 4.1 that
\[ (2) \quad \omega_W = \omega_W^g, \]
the desired conclusion. \(\Box\)

**Remark 4.9.** The above proof actually establishes equality of periods of \(\omega_W\) and \(\omega_W^g\) without using harmonicity. This will recur in the proof of Theorem 7.1.

An alternate interpretation of Lemma 4.8 is to look at the harmonic map \(f_\omega : \tilde{S} \to \mathbb{R}\) that \(\omega\) induces by integrating along paths. Uniqueness of the harmonic form \(\omega\) in its cohomology class coupled with triviality of the pseudo-action shows that the harmonic map \(f_\omega : \tilde{S} \to \mathbb{R}\) is canonical, i.e. it does not change under passing to a finite index subgroup of \(\Gamma\). In further generalizations of Lemma 4.8 (particularly in the proof of Theorem 7.1), this interpretation will be more useful.

**The commensurator of a form:**

The notion of the commensurator of a form will be essential for producing \(\Gamma\)-discrete patterns. Cohomology with compact supports will be denoted by \(H_p^c(\cdot)\).

**Definition 4.10.** Let \(\Gamma \ll G\) be a lattice in a semi-simple Lie group \(G\) with associated symmetric space \(X\), and let \(S = X/\Gamma\). Let \(\omega\) be a closed form such that \([\omega] \in H^p(S, \mathbb{Q})\) or \([\omega] \in H^p_c(S, \mathbb{Q})\) is a non-zero cohomology class. Let \(p : X \to S\) denote the universal cover. The commensurator \(\text{Comm}(\omega)\) of the form \(\omega\) is defined as
\[ \text{Comm}(\omega) = \{ h \in G \mid h^* p^* \omega = p^* \omega \}. \]

A subgroup \(H\) of \(G\) is said to commensurate \(\omega\) if \(H \ll \text{Comm}(\omega)\).
We shall first give a proof of the discreteness of \( \text{Comm}(\omega) \) specialized to surfaces, as it makes the geometry of the situation clear.

**Proposition 4.11.** Let \( \Gamma < \text{PSL}_2(\mathbb{R}) \) be a cocompact torsion-free lattice and \( S = \mathbb{H}^2/\Gamma \) be the quotient surface. Then for any non-zero harmonic \( \omega \) representing an element of \( H^1(S, \mathbb{Q}) \), the group \( \text{Comm}(\omega) \) is discrete.

**Proof.** The existence of a unique harmonic form \( \omega \) in the cohomology class \([\omega] \in H^1(S, \mathbb{Q})\) is guaranteed by Theorem 4.1. Let \( S_0 \) denote the non-empty set of zeros of \( \omega \). We remark that any harmonic form is the real part of a holomorphic 1-form (an abelian differential) and hence \( S_0 \) is finite.

Let \( S \) denote the lift of \( S_0 \) to the universal cover \( \mathbb{H}^2 \). Then \( S \) is discrete. Since \( \text{Comm}(\omega) \) commensurates \( \omega \), its action on \( \mathbb{H}^2 \) preserves \( S \). Hence \( \text{Comm}(\omega) \) is discrete, by Lemma 4.5. \( \Box \)

**Proposition 4.12.** Let \( \Gamma < \text{PSL}_2(\mathbb{R}) \) be a non-uniform torsion-free lattice such that the quotient hyperbolic surface \( S = \mathbb{H}^2/\Gamma \) has genus greater than zero. Then for any non-zero \( L^2 \)-harmonic \( \omega \in H^1_{(2)}(S, \mathbb{Q}) \), the group \( \text{Comm}(\omega) \) is discrete.

**Proof.** The existence of a unique \( L^2 \) harmonic form \( \omega \) in the cohomology class \([\omega] \in H^1_{(2)}(S, \mathbb{Q})\) is guaranteed by Theorem 4.2. If the zero set of \( \omega \) is non-empty, then exactly the same proof as that of Proposition 4.11 shows that \( H \) is discrete.

Let \( \overline{S} \) denote the Riemann surface compactification (technically the Baily–Borel–Satake compactification) of \( S \), obtained by adding a point at infinity for every cusp of \( S \). Let \( C \) denote the collection of points thus added. If the zero set of \( \omega \) is empty, then its zeros and poles (equivalently, the zeros and poles of the associated holomorphic differential) necessarily lie in \( C \). Then \( \omega \) gives rise to a foliation \( \mathcal{F} \) on \( S \) in terms of the integral curves of the vector field which pointwise realizes the kernel of \( \omega \). There is a distinguished collection of leaves of \( \mathcal{F} \) connecting points of \( C \) to points of \( C \). We call this set the collection of *saddle connections*, and denote it by \( \mathcal{J}_0 \). Let \( \mathcal{J} \) denote the lift of \( \mathcal{J}_0 \) to \( \mathbb{H}^2 \). Then \( \mathcal{J} \) is a discrete pattern of geodesics in \( \mathbb{H}^2 \) preserved by \( \text{Comm}(\omega) \). Again \( \text{Comm}(\omega) \) is discrete, by Lemma 4.5. \( \Box \)

Finally, we prove a general theorem that works for all rank one Lie groups. We direct the reader to [66, 1], from which the main idea used in the following Proposition is taken:

**Proposition 4.13.** Let \( X = \mathbb{H}^n \) or \( \mathbb{C} \mathbb{H}^n \). For \( \Gamma \) a torsion-free lattice, let \( S = X/\Gamma \). Let \( \omega \) be a non-zero harmonic or \( L^2 \)-harmonic 1-form according as \( S \) is compact or non-compact. Then \( \text{Comm}(\omega) \) is discrete.

**Proof.** Let \( p : X \rightarrow S \) denote the universal cover. We now argue by contradiction. Suppose that \( \text{Comm}(\omega) \) is not discrete. Since the associated Lie group \( G \) (i.e. \( \text{SO}(n, 1) \) or \( \text{SU}(n, 1) \)) is simple, it follows that \( \text{Comm}(\omega) \) is dense in \( G \), as \( \text{Comm}(\omega) \) contains the Zariski dense subgroup \( \Gamma \). Also, since \( \text{Comm}(\omega) \) preserves \( p^* (\omega) \), we have that \( G \) must preserve \( p^* (\omega) \), since \( G \) is identified with the group of isometries of \( X \). That is, \( p^* (\omega) \) is a \( G \)-invariant non-zero harmonic 1-form on \( X \). Hence \( p^* (\omega) \) gives a non-zero harmonic differential 1-form \( \omega^* \) on the compact dual [66, 1] (of \( \mathbb{H}^n \) or \( \mathbb{C} \mathbb{H}^n \)). Since the compact duals of \( \mathbb{H}^n \) and \( \mathbb{C} \mathbb{H}^n \) have trivial first cohomology, this is a contradiction. \( \Box \)

### 4.3. Commensurations of forms and quotient groups

We will need a mild generalization of the results above to forms arising from quotients of a lattice \( \Gamma \). We let \( Q = \Gamma / K \) be a quotient. Note that if \( H_1(Q, \mathbb{Q}) \neq 0 \) then we immediately have that \( H^1(Q, \mathbb{Q}) \neq 0 \). Since \( H_1(Q, \mathbb{Q}) \) is a quotient of \( H_1(\Gamma, \mathbb{Q}) \), it is clear what is meant by the homological pseudo-action of an element
$g \in \text{Comm}_G(\Gamma)$ on $H_1(Q, \mathbb{Q})$, though as before this is not an honest action. Observe however that the set of elements $g \in G$ whose pseudo-action on $H_1(Q, \mathbb{Q})$ is trivial forms a monoid, and that by duality this is the monoid with trivial cohomological pseudo-action on $H^1(Q, \mathbb{Q})$. We have the following fact:

**Lemma 4.14.** Let $Q = \Gamma/K$, suppose $H_1(Q, \mathbb{Q}) \neq 0$, and let $\text{Comm}_{pa}(\Gamma, K)$ denote a maximal subgroup of the monoid of elements which have a trivial pseudo-action on $H_1(Q, \mathbb{Q})$. If there exists a nontrivial $L^2$ harmonic form on $X/T$ representing a pullback of a cohomology class of $Q$, then $\text{Comm}_{pa}(\Gamma, K)$ is discrete.

Here we could assume that $\text{Comm}_{pa}(\Gamma, K)$ contains $K^G$ as is strictly prescribed by the definition, though for the proof this assumption is not necessary.

**Proof of Lemma 4.14.** Let $\omega \in H^1(Q, \mathbb{Q})$ be a nontrivial cohomology class. Then the quotient map $q : \Gamma \to Q$ induces a pullback form $q^*\omega \in H^1(Q, \mathbb{Q})$ as well as an induced map $q_* : H_1(\Gamma, \mathbb{Q}) \to H_1(Q, \mathbb{Q})$. If $\sigma$ is any 1–cycle on $X/T$ then by definition

$$\int_{\sigma} q^*\omega = \omega(q_*\sigma).$$

If we write $p^* q^*\omega$ for the form on $X$ given by pullback under the covering map $p : X \to X/\Gamma$, we have that $q^* p^* q^*\omega = p^* q^*\omega$ for all $g \in \Gamma_1$ by an argument identical to that in Lemma 4.8, whence the form $p^* q^*\omega$ is invariant under $\text{Comm}_{pa}(\Gamma, K)$.

Since there exists a nontrivial $L^2$ harmonic form on $X/T$ representing a pullback of a cohomology class of $Q$, then we may apply Proposition 4.13 to conclude that $\text{Comm}_{pa}(\Gamma, K)$ is discrete. □

## 5. Abelian Quotients and Harmonic 1-Forms

We are now in a position to assemble the pieces to prove Theorem 1.4 in the case where the first Betti number of $Q$ is positive—a hypothesis contained in item 1 of that result. To give the reader geometric intuition, we will prove the results first for $\text{PSL}_2(\mathbb{R})$ and then for general $G$, since in the case of $\text{PSL}_2(\mathbb{R})$ the arguments tend to be more concrete and geometrically transparent. Further, the argument naturally splits into two cases:

1. The vanishing cuspidal case, amenable to $L^2$–cohomology techniques. For $\text{PSL}_2(\mathbb{R})$, this is the case where the underlying hyperbolic surface has genus greater than zero.
2. The non-vanishing cuspidal case, where discrete patterns of horoballs are used to obtain discreteness of the commensurator (see Theorem 5.7). For $\text{PSL}_2(\mathbb{R})$, this is the case where the underlying hyperbolic surface has genus equal to zero: $L^2$–cohomology vanishes.

We shall, in the interests of concreteness, first concentrate on the case $K = [\Gamma, \Gamma]$, only later addressing the general case with $b_1(Q) > 0$. We shall write $S = X/\Gamma$ to denote the quotient locally symmetric space for both $X = \mathbb{H}^2$ or a general rank one symmetric space $X$.

### 5.1. The non-vanishing cuspidal case

We first deal with the case where the map $H^1(S, \mathbb{Q}) \to H^1(\partial S, \mathbb{Q})$ is not injective; equivalently $H_1(\partial S, \mathbb{Q}) \to H_1(S, \mathbb{Q})$ is not surjective. In more sophisticated language, this is the case where the Eichler–Shimura cohomology is non-vanishing. We will first deal with the manifold case (i.e. where $\Gamma$ is torsion-free), relegating the general orbifold case to the sequel. We begin with an illustrative special case, where $G = \text{PSL}_2(\mathbb{R})$.

**Theorem 5.1.** Let $\Gamma < \text{PSL}_2(\mathbb{R})$ be a torsion-free arithmetic lattice (uniform or non-uniform) such that the quotient hyperbolic surface $S = \mathbb{H}^2/\Gamma$ has genus greater than zero. Then the commensurator of $[\Gamma, \Gamma]$ in $\text{PSL}_2(\mathbb{R})$ is discrete.
Proof. Let $H = [\Gamma, \Gamma]$. By Proposition 2.3, $\text{Comm}_G H < \text{Comm}_G \Gamma$. Let $S = \mathbb{H}^2 / \Gamma$. For $h \in \text{Comm}_G H$, consider the pseudo-action of $\Gamma^h$ on $H^1(S, \mathbb{Q})$ furnished by Corollary 3.8 and Lemma 4.8. By Corollary 3.7, we may conclude that the pseudo-action is trivial for any such $h$. It follows from Lemma 4.8 that for any $[\omega] \in H^1_{(2)}(S, \mathbb{Q})$, the group $\Gamma^h$ commensurates the unique harmonic representative $\omega_{\text{harm}} \in [\omega]$. Again, since the pseudo-action of $\Gamma$ on $H^1(S, \mathbb{Q})$ is trivial, it follows that the group

$$\Gamma := \langle \Gamma^h \mid h \in \text{Comm}_G H \rangle$$

generated by $\{\Gamma^h \mid h \in \text{Comm}_G H\}$ commensurates $\omega_{\text{harm}}$. By Propositions 4.11 and 4.12, $\Gamma$ is discrete.

Since $\Gamma$ contains $\Gamma$, it is a lattice. By Corollary 3.5, we have $\Gamma^h = \Gamma$ for all $h \in \text{Comm}_G H$. It follows that $\text{Comm}_G H$ normalizes $\Gamma$. The result now follows from Lemma 2.2.

Theorem 5.1 can be easily generalized to arithmetic lattices in rank one semi-simple Lie groups.

**Theorem 5.2.** Let $G = \text{SO}(n, 1)$ or $\text{SU}(n, 1)$ and let $X$ denote the associated symmetric space. Let $\Gamma < G$ be a torsion-free arithmetic lattice (uniform or non-uniform) such that the quotient manifold $S = X/\Gamma$ satisfies

1. $H^1(S, \mathbb{Q}) \neq 0$ if $S$ is compact.
2. $H^1_{(2)}(S, \mathbb{Q}) \neq 0$ if $S$ is non-compact.

Then the commensurator of $[\Gamma, \Gamma]$ in $G$ is discrete.

**Proof.** As in the proof of Theorem 5.1, let $H = [\Gamma, \Gamma]$. Proposition 2.3 gives $\text{Comm}_G H < \text{Comm}_G \Gamma$. Let $h \in \text{Comm}_G H$. By Corollary 3.4, we have that the homological pseudo-action of $h$ on $H_1(S, \mathbb{Q})$ is trivial. Lemma 4.8 now shows that for any $[\omega] \in H^1(S, \mathbb{Q})$ or $[\omega] \in H^1_{(2)}(S, \mathbb{Q})$ when $S$ is compact and non-compact, respectively, the group $\Gamma^h$ commensurates the unique harmonic representative $\omega_{\text{harm}} \in [\omega]$. It follows from Proposition 4.13 that $\text{Comm}(\omega_{\text{harm}})$ is discrete. Hence as in the proof of Theorem 5.1,

$$\Gamma := \langle \Gamma^h \mid h \in \text{Comm}_G H \rangle$$

is discrete. Since $\Gamma$ contains $\Gamma$, it is a lattice.

Since there are only finitely many subgroups of $\Gamma$ of index $[\Gamma : \Gamma]$ and since $\Gamma^h < \Gamma$ by the definition of $\Gamma$, it follows that there exists $N \in \mathbb{N}$ such that for every $h \in \text{Comm}_G H$, the element $h^N$ normalizes $\Gamma$. Lemma 2.1 now shows that $\text{Comm}_G H$ is discrete. \hfill $\square$

### 5.2. The vanishing cuspidal case for $\text{PSL}_2(\mathbb{R})$

We would now like to relax the hypothesis of positive genus in Theorem 5.1. This is the case where the map $H^1(S, \mathbb{Q}) \to H^1(\mathbb{H}^2, \mathbb{Q})$ is injective; equivalently $H_1(\mathbb{H}^2, \mathbb{Q}) \to H_1(S, \mathbb{Q})$ is surjective. The proof technique here is quite different and borrows from [39]. For now, we give the statement and proof only in the case $G = \text{PSL}_2(\mathbb{R})$, and we will address the the general case in Theorem 5.7 below.

**Theorem 5.3.** Let $\Gamma < \text{PSL}_2(\mathbb{R})$ be a torsion-free lattice such that the quotient hyperbolic surface $S = \mathbb{H}^2 / \Gamma$ has genus equal to zero. Then the commensurator of $[\Gamma, \Gamma]$ in $\text{PSL}_2(\mathbb{R})$ is discrete.

**Proof.** Note that since $\Gamma$ is torsion-free then it must be a surface group or a free group. Since $S$ has genus zero, we have that $\Gamma$ must therefore be non-uniform, whence it must be commensurable (in the wide sense) with $\text{PSL}_2(\mathbb{Z})$ (see Theorem 5.2 of [48]). Replacing $\Gamma$ by a conjugate, we may thus assume that $\Gamma$ is commensurable with $\text{PSL}_2(\mathbb{Z})$.

Let $H = [\Gamma, \Gamma]$. By Proposition 2.3,

$$\text{Comm}_G H < \text{Comm}_G \Gamma = \text{PGL}_2(\mathbb{Q}).$$
We shall again consider the pseudo-action of $\Gamma^h$ on $H^1(S, \mathbb{Q})$ furnished by Corollary 3.8. Since $S$ is of genus zero, $H_1(S, \mathbb{Q})$ is generated by the homology classes of the cusps. Since $\Gamma$ is torsion-free, we have that $H_1(S, \mathbb{Q})$ has rank at least two, which implies that the homology class of each cusp is nontrivial, and that no two distinct cusps represent the same homology class. If $\Gamma^h \subset \text{PSL}_2(\mathbb{Z})$, then by Corollary 3.5, $\Gamma^h = \Gamma$, forcing $h \in \text{PSL}_2(\mathbb{Z})$. We therefore assume that $\Gamma^h$ is not contained in $\text{PSL}_2(\mathbb{Z})$. As in the proof of Corollary 3.3, it now suffices to show that the homology pseudo-action of $\Gamma^h$ on $H_1(S, \mathbb{Q})$ is non-trivial.

The proof is now a reworking in this context of [39, Section 4.2]. We shall argue by contradiction, showing that if the homology pseudo-action of $\Gamma^h$ on $H_1(S, \mathbb{Q})$ is trivial then $\Gamma^h \subset \text{PSL}_2(\mathbb{Z})$, contrary to our assumption. So, suppose that that the homology pseudo-action of $\Gamma^h$ on $H_1(S, \mathbb{Q})$ is trivial. Let $\gamma \in \Gamma$ be a parabolic element fixing infinity. Let $y \in \Gamma^h$ be not contained in $\text{PSL}_2(\mathbb{Z})$, and suppose that the $y$-pseudo-action on $H_1(S, \mathbb{Q})$ is trivial. Then there exists a positive integer $n$ such that $[(\gamma^n)^y] = [\gamma^n]$, where $[\ ]$ denotes the corresponding homology class in $S = \mathbb{H}^2/\Gamma$. Thus $(\gamma^n)^y$ represents a power of a free homotopy class of the cusp of $S$ given by $\gamma$; and so there exists $r \in \Gamma$ such that $((\gamma^n)^y)^r = (\gamma^n)^yr$ is a parabolic element fixing infinity, i.e. $yr \in \text{PSL}_2(\mathbb{Q})$ is of the form

$$yr = \begin{pmatrix} \lambda & t \\ 0 & \lambda^{-1} \end{pmatrix}$$

where $\lambda, t \in \mathbb{Q}$.

Since

$$[(\gamma^N)^{yr}] = [(\gamma^N)^y] = [\gamma^N],$$

it follows by an easy computation that $\lambda = 1$ (see [39, Section 4.2] for instance), so that

$$yr = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

where $t \in \mathbb{Q}$.

Repeating the above computation for the opposite parabolic in $H$ fixing 0, there exists $q \in \Gamma$ such that

$$yq = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix},$$

where $s \in \mathbb{Q}$.

We thus conclude that

$$r^{-1}q = (yr)^{-1}yq \in \Gamma$$

is of the form

$$\begin{pmatrix} 1 - ts & -t \\ s & 1 \end{pmatrix},$$

forcing $s, t$ to lie $\mathbb{Z}$. Hence $y \in \text{PSL}_2(\mathbb{Z})$. Since $y$ is an arbitrary element of $\Gamma^h$, it follows that $\Gamma^h$ is contained in $\text{PSL}_2(\mathbb{Z})$, the desired contradiction. \hfill $\Box$

5.3. Orbifolds. An orbifold is said to be good if it is finitely covered by a manifold. In this section we shall be interested in (not necessarily torsion-free) lattices $\Gamma \subset G$ with $G$ semi-simple. The orbifold $S = X/\Gamma$ is then necessarily good by Selberg’s Lemma. Let $S_x$ denote the singular set of $S$. We shall say that a form $\omega$ on $S \setminus S_x$ is harmonic (resp. $L^2$-harmonic) if there is a Galois orbifold cover $q : M \rightarrow S$ of $S$ such that

(1) $M$ is a manifold

(2) there exists a smooth harmonic (resp. $L^2$-harmonic) form $\omega_M$ invariant under the finite deck group such that $q^* (\omega) = \omega_M$ on $M \setminus q^{-1}(S_x)$. 

We shall say that an orbifold $S = X/\Gamma$ has $b_1(S) > 0$ if $b_1(\Gamma) > 0$, i.e. if the abelianization of $\Gamma$ has positive rank. Usual Hodge theory then generalizes in a straightforward way to show that if $S$ is a compact good orbifold with $b_1(S) > 0$, then $S$ admits a non-zero harmonic 1–form. If $S$ is non-compact, but modeled on $X = \mathbb{H}^n$ or $\mathbb{C} \mathbb{H}^n$, then again the Hodge theorem for $L^2$–cohomology generalizes in a straightforward way. We first indicate the generalization of Theorem 5.2 to orbifolds.

**Theorem 5.4.** Let $G = SO(n, 1)$ or $SU(n, 1)$ and let $X$ denote the associated symmetric space. Let $\Gamma < G$ be a (not necessarily torsion-free) arithmetic lattice (uniform or non-uniform) such that the quotient orbifold $S = X/\Gamma$ satisfies

1. $H^1(S, \mathbb{Q}) \neq 0$ if $S$ is compact.
2. $H^1_{(2)}(S, \mathbb{Q}) \neq 0$ if $S$ is non-compact.

Then the commensurator of $[\Gamma, \Gamma]$ in $G$ is discrete.

**Proof.** For $H = [\Gamma, \Gamma]$, Proposition 2.3 gives $\text{Comm}_G H < \text{Comm}_G \Gamma$ as before. Let $h \in \text{Comm}_G H$. Theorem 3.4 allows us to reduce to the case that the homological pseudo-action of $h$ on $H_1(S, \mathbb{Q})$ is trivial. Further, the proof of Lemma 4.8 allows us to pass to a common manifold cover of $S$ and $S^h$. The hypothesis of the present theorem guarantees the existence of a non-zero harmonic (resp. $L^2$–harmonic) $\omega_{\text{harm}}$ on $S$ compact (resp. non-compact). The group $\Gamma^h$ commensurates $\omega_{\text{harm}}$, by Lemma 4.8. Proposition 4.13 now shows that $\text{Comm}(\omega_{\text{harm}})$ is discrete. The rest of the proof is identical to that of Theorem 5.2.

5.3.1. **Hyperbolic 2–orbifolds.** We now indicate, as a corollary, the generalization of Theorem 5.1 to 2–dimensional hyperbolic orbifolds $S$. To begin, we shall need to recall a basic implication of the Zucker conjecture in the case of a non-compact 2–dimensional hyperbolic orbifold of finite volume. We refer the reader to [12, 70, 25] for proofs and for more detail. In this special case, half the $L^2$–Betti number coincides exactly with (the usual topological) genus of the underlying topological surface $S$, i.e. $\frac{b_1(\mathbb{H}^2)}{2} = g(S)$. In general, the Zucker conjecture identifies $b_1^{(2)}$ with the usual $b_1$ of the Baily–Borel–Satake compactification, which in the case of a hyperbolic surface of finite volume is exactly the completion of the underlying Riemann surface.

**Corollary 5.5.** Let $\Gamma < \text{PSL}_2(\mathbb{R})$ be a (not necessarily torsion-free) arithmetic lattice (uniform or non-uniform) such that the genus $g(S_1)$ satisfies $g(S_1) > 0$, where $S_1$ denotes the topological manifold underlying $\mathbb{H}^2/\Gamma$. Then the commensurator of $[\Gamma, \Gamma]$ in $\text{PSL}_2(\mathbb{R})$ is discrete.

**Proof.** The compact case is just a special case of Case 1 of Theorem 5.4. For the non-compact case, the inequality $g(S_1) > 0$ is exactly equivalent to the inequality $b_1^{(2)} > 0$, by the remarks preceding the present corollary. The non-compact case is now a special case of Case 2 of Theorem 5.4.

It remains to generalize the vanishing cuspidal case (Theorem 5.3) to orbifolds. We shall give a more geometric proof in the general context of a rank 1 symmetric space (Theorem 5.7) thereby uncovering the geometry behind the computation in the proof of Theorem 5.3. For now we only sketch the modifications needed for hyperbolic orbifolds of dimension two. The proof of Theorem 5.3 goes through as before when the orbifold $S$ has at least three cusps, since in this case $H_1(S, \mathbb{Q})$ is generated by the homology classes of the cusps, each of which is nontrivial and no two distinct cusps represent the same homology class. The rest of the computation in the proof of Theorem 5.3 goes through as before using the triviality of the homology pseudo-action.

If $S$ is of genus zero and has exactly one cusp, then the homology class carried by the cusp is necessarily either trivial or torsion, and so $b_1(\Gamma) = 0$. It thus remains only to deal with the case that
Claim 5.8. \( \Gamma^h < \Gamma_1 \).
We complete the proof modulo Claim 5.8. Since there are only finitely many subgroups of \( \Gamma \) of index \([\Gamma : \Gamma] \) and since \( \Gamma^h < \Gamma \) by Claim 5.8, it follows that there exists \( N \in \mathbb{N} \) such that for every \( h \in \text{Comm}_G^0H \), the element \( h^N \) normalizes \( \Gamma \). Lemma 2.1 now shows that \( \text{Comm}_G^0H \) is discrete. The result now follows.

**Proof of Claim 5.8:** By Corollary 3.6, we may assume that the homology pseudo-action of \( \Gamma^h \) on \( H_1(S, \mathbb{Q}) \) is trivial. Let \( \gamma \in \Gamma \) be a parabolic element representing \( z \in H_1(T, \mathbb{Q}) \), and such that \( i_*\gamma(z) \) is non-zero. Suppose that \( \gamma \) fixes \( x \in \partial X \). Let \( y \in \Gamma^h \), and suppose that the \( y \)-pseudo-action on \( H_1(S, \mathbb{Q}) \) is trivial. Then there exists a positive integer \( n \) such that \n
\[
[(\gamma^n)^y] = [\gamma^n] = nz,
\]

where \([. .] \) denotes the corresponding homology class. Thus \((\gamma^n)^y\) represents a power of the homology class \( z \) in \( H_1(T, \mathbb{Q}) \) given by \( \gamma \); and so there exists \( r \in \Gamma \) such that \((\gamma^n)^y)^r = (\gamma^n)^yr\) is a parabolic element of \( G \) fixing \( x \). Let \( G_x < G \) denote the parabolic subgroup of \( G \) fixing \( x \). Then \( yr \in G_x \).

Any element of the parabolic subgroup \( G_x \) can be decomposed as \( A_\lambda N_\lambda \), where \( A_\lambda \) acts on \( \partial X \setminus \{x\} \) by a conformal homothety and \( N_\lambda \) acts by an isometry. For \( X = \mathbb{H}^n \), these are all Euclidean similarities and for \( X = \mathbb{C}\mathbb{H}^n \), these are Heisenberg similarities (see [60, Section 8.1]). Let \( A_\lambda(yr) > 0 \) denote the scale factor of the homothety component of \( yr \). Let \( H_x \in \mathcal{H} \) denote the horoball in \( X \) based at \( x \). Since \n
\[
[(\gamma^n)^{yr}] = A_\lambda(yr)[\gamma^n],
\]

the scale factor \( A_\lambda(yr) \) must equal one. But \( A_\lambda(yr) = 1 \) if and only if \( yr \) preserves the horosphere \( \partial H_x \). Since \( r \in \Gamma \) necessarily preserves \( \partial H_x \), it follows that \( y \) stabilizes \( \partial H_x \), i.e., \( y \in \Gamma_1 \). Since \( y \) was arbitrary, this completes the proof of Claim 5.8.

Combining Theorems 5.2 and 5.7 we now have:

**Theorem 5.9.** Let \( \Gamma < G \) be a (not necessarily torsion-free) irreducible arithmetic lattice (uniform or non-uniform) in a semi-simple Lie group \( G \). Then the commensurator of \([\Gamma, \Gamma] \) in \( G \) is discrete if and only if \( b_1(\Gamma) > 0 \).

**Proof.** The theorem follows from Theorems 5.2 and 5.7 and the fact that irreducible lattices in all other semi-simple Lie groups have \( b_1(\Gamma) = 0 \).}

5.4. **The case** \( b_1(Q) > 0 \). We now generalize the discussion of this section to establish the following result.

**Theorem 5.10.** Let \( \Gamma < G \) be a (not necessarily torsion-free) lattice (uniform or non-uniform) in a rank one simple Lie group. Let \( K < \Gamma \) be an infinite normal subgroup, and let \( Q = \Gamma/K \). If \( b_1(Q) > 0 \) then \( \text{Comm}_G(K) \) is discrete.

**Proof.** Recall that we write \n
\[
K^G = \{\langle K^g \rangle \mid g \in \text{Comm}_G(K)\}
\]

for the subgroup generated by \( K^g \) as \( g \) ranges over \( \text{Comm}_G(K) \). By Theorem 3.13, we have that if \( z \in K^G \) then the pseudo-action of \( z \) on \( Q \) is trivial.

Let \( \text{Com}_{pa}(\Gamma, K) \) denote a maximal subgroup of \( \text{Com}_{pa}(\Gamma, K) \subset G \) containing \( K^G \) whose pseudo-action on \( H_1(Q, \mathbb{Q}) \) is trivial. By hypothesis, \( H^1(Q, \mathbb{Q}) \neq 0 \). Suppose first that there exists an \( L^2 \) harmonic form on \( X/\Gamma \) representing a pullback of a nontrivial class in \( H^1(Q, \mathbb{Q}) \). In this case, \( \text{Com}_{pa}(\Gamma, K) \) is discrete by Lemma 4.14. Note that \( \text{Com}_{pa}(\Gamma, K) \) contains \( \Gamma \) since \( K \) is normalized by \( \Gamma \), and also that \( \text{Com}_{pa}(\Gamma, K) \) contains \( K^G \) by definition. That \( \text{Comm}_G(K) \) is discrete now follows from Proposition 3.15.
Otherwise, we are in the vanishing cuspidal case. Recall the notation $S = X/\Gamma$. Writing $q: \Gamma \to Q$ for the quotient map, we have that

$$q_* \circ i_* : H_1(\partial S, \mathbb{Q}) \to H_1(Q, \mathbb{Q})$$

is surjective (where $i: \partial S \to S$ denotes inclusion). As in the proof of Theorem 5.7, there exists a cusp $T$ of $S$ and $z \in H_1(T, \mathbb{Q})$ such that

$$q_* \circ i_*(z) \neq 0.$$

Again, as in the proof of Theorem 5.7, let $\Gamma_1$ be the over-lattice of $\Gamma$ preserving the collection $\mathcal{H}$ of horoballs corresponding to lifts of $T$ to $X$.

If $g \in \text{Comm}_G(K)$, we then have that the homology pseudo-action of $K^g$ on $H_1(Q, \mathbb{Q})$ is trivial. In particular, the homology pseudo-action of $K^g$ on $z$ is trivial. It follows as in Claim 5.8 that $K^g$ is contained in $\Gamma_1$ for all $g \in \text{Comm}_G(K)$. Hence $K^G$ is discrete. Proposition 3.15 now implies that $\text{Comm}_G(K)$ itself is discrete. \hfill \Box

We conclude this section by giving two sets of examples to which Theorem 5.10 applies.

**Irrational pencils in complex hyperbolic manifolds**: Many cocompact arithmetic lattices in $SU(2, 1)$ admit irrational pencils, i.e. $S = X/\Gamma$ admits a holomorphic fibration (with singular fibers) onto a Riemann surface of genus greater than zero. Let $F$ denote the general fiber and $i : F \to S$ denote inclusion. Then $K = i_*(\pi_1(F))$ is normal in $\Gamma$ and $Q = \Gamma/K$ is a surface group. Theorem 5.10 applies to show that $\text{Comm}_G(K)$ is discrete. We note that M. Kapovich in unpublished work [36] (see [7] for a small generalization) established that $K$ is never finitely presented.

**Real hyperbolic manifolds that algebraically fiber**: Agol [2] shows that hyperbolic 3-manifolds virtually fiber over the circle with surface group fibers. The resulting normal surface subgroups were dealt with in [45] without the arithmeticity hypothesis. However, a new family of examples of finitely generated (but not necessarily finitely presented) normal subgroups of arithmetic hyperbolic $n$-manifolds has recently been discovered. A classical result of Dodziuk [17, 3] shows that the first $L^2$-betti number of a hyperbolic manifold of dimension greater than 2 vanishes. Kielak [37] shows that a cubulated hyperbolic group $\Gamma$ is virtually algebraically fibered (i.e. $\Gamma$ admits a virtual surjection to $Z$ with a finitely generated kernel) if and only if $b_1^1(\Gamma) = 0$. On the other hand, Bergeron-Haglund-Wise [6] show that standard cocompact arithmetic congruence subgroups $\Gamma$ of $SO(n, 1)$ are cubulated. Thus standard cocompact arithmetic congruence subgroups $\Gamma$ of $SO(n, 1)$ admit finitely generated normal subgroups $K$ with quotient $Z$. This furnishes a family of examples $K$ to which Theorem 5.10 applies to show that $\text{Comm}_G(K)$ is discrete (since $b_1(Q) = b_1(Z) = 1$ in this case).

6. **Non-abelian quotients and harmonic maps**

For the purposes of this section, $G$ will be a rank one semi-simple Lie group, $X$ the associated symmetric space, $\Gamma \subset G$ an arithmetic lattice and $S = X/\Gamma$. The key point in Section 4 was to find canonical representatives of rational cohomology classes that are invariant under the homology pseudo-action (Lemma 4.8). Rational cohomology classes arise from abelian representations from $\Gamma(= \pi_1(S))$ to $\mathbb{Q}$. The canonical representatives were given by harmonic 1-forms in Section 4. In this section, we shall use analogous results from non-abelian representations. The relevant background comes from non-abelian Hodge theory developed by Hitchin [32], Donaldson [18], Corlette [14], Simpson [65, 64] and others. The main output of the theory we shall extract is the existence of a
canonical harmonic map into a symmetric space of non-compact type. We extract the property of canonical maps we need via the following definition:

**Definition 6.1.** Let $G$ be a semi-simple Lie group and let $X$ be the associated symmetric space. Let $\Gamma < G$ be a discrete subgroup. Let $\rho : \Gamma \to L$ be a Zariski dense representation from $\Gamma$ to a semi-simple Lie group $L$. Let $Y$ be the symmetric space of $L$. An association $\rho \mapsto f_{\rho}$ of a $(\Gamma, \rho)$–equivariant map $f_{\rho} : X \to Y$ is said to be canonical if it is invariant under passing to finite index subgroups of $\Gamma$. That is, if $\Gamma_1 < \Gamma$ is a finite index subgroup and if $\rho_1 : \Gamma_1 \to L$ is a representation such that $\rho_1$ agrees with $\rho$ on $\Gamma_1$, then the induced map $f_{\rho_1}$ is equal to $f_{\rho}$ on $X$.

6.1. **Existence of canonical harmonic maps.** We caution the reader at the outset that the canonical harmonic maps, whose existence and uniqueness we recall here, are not always of finite energy; it suffices for our purposes that they are canonical, i.e. unique given certain conditions as per Definition 6.1. We start with the following theorem due to Corlette [14] andLabourie [44]:

**Theorem 6.2.** [14, 44] Suppose that $S = X/\Gamma$ is a compact orbifold. Let $\rho : \Gamma \to L$ be a Zariski dense representation from $\Gamma$ to a semi-simple Lie group $L$. Let $Y$ be the symmetric space of $L$. Then there is a unique $(\Gamma, \rho)$–equivariant harmonic map $h_{\rho} : X \to Y$.

6.1.1. **Non-uniform lattices: non-exceptional cases.** We shall need an analogous theorem for non-compact finite volume orbifolds. The theory here splits into finite and infinite energy harmonic maps. The existence of a finite energy equivariant harmonic map is guaranteed provided that a finite energy equivariant map exists. Finite energy maps were constructed by Corlette [15] following an idea of Mok [55] in the context of quaternionic and octonionic hyperbolic space. The same idea was extended to $G = SU(n, 1)$, wherein we have $X = \mathbb{CH}^n$, for $n > 2$ by Koziarz and Maubon [42]. We describe here the analogous construction for real hyperbolic space.

Thus, let $X = \mathbb{H}^n$ and $S = X/\Gamma$ be non-compact. For each cusp of $X$, choose a neighborhood $\mathcal{C}$ bounded by an embedded flat Euclidean orbifold $\mathcal{T}$. Thus, $\mathcal{T}$ is a compact quotient of a horosphere. Let $S^0$ denote a compact core of $S$ obtained by removing the interiors of the neighborhoods $\mathcal{C}$. Each $\mathcal{C}$ is foliated by ‘vertical’ geodesics. We define a retraction $r : S \to S^0$ by mapping a point $x \in S \setminus S^0$ on such a vertical geodesic in $\mathcal{C}$ to the unique point of $\mathcal{T}$ that the geodesic meets, and by setting $r(x) = x$ if $x \in S^0$.

Suppose that we are given a Zariski dense representation $\rho : \Gamma \to L$ from $\Gamma$ to a semi-simple Lie group $L$. Recall that we denote by $Y$ the associated symmetric space of $L$. To prove the existence of a finite energy equivariant map from $X$ to $Y$, it suffices to show that $r$ has finite energy. Let $\mathcal{T}_t$ denote the collection of points in $\mathcal{C}$ at distance $t$ from $\mathcal{T}$. The volume of $\mathcal{T}_t$ is $e^{-(n-1)t}$, after normalizing suitably. Now, the norm of the map $r$ at any point of $\mathcal{T}_t$ in any direction is $e^t$. Thus the energy of $r$ is, up to a scalar, given by

$$E^2(r) = \int_0^\infty (n-1)e^{2t}e^{-(n-1)t} dt,$$

which clearly is finite if and only if $n > 3$.

Thus, unless

$$G \in \{\text{SO}(2, 1), \text{SO}(3, 1), \text{SU}(2, 1)\},$$

retracting $S$ onto a compact core $S^0$ via $r$ and mapping $S^0$ over to $Y$ in a way which is $(\Gamma, \rho)$–equivariant, we obtain a finite energy $(\Gamma, \rho)$–equivariant map from $X$ to $Y$. By a standard application of the heat flow, we can propagate this map defined on $\hat{\mathcal{S}}$ to a finite energy $(\Gamma, \rho)$–equivariant harmonic map defined on $\hat{\mathcal{S}} = X$. Precisely, we apply the following result which can be found in [15] and [43]:
**Theorem 6.3** (See [15], Theorem 2.2 and and [43]). Suppose $M$ is a complete Riemannian manifold and $N$ a complete, simply connected manifold with non-positive sectional curvature. Suppose

$$\rho: \pi_1(M) \to \text{Isom}(N)$$

induces a fixed point free action on $\partial N$. If there exists a $\rho$–equivariant map from $M$ to $N$ which has finite energy, then there exists a harmonic $\rho$–equivariant map from $M$ to $N$ with finite energy.

It is easy to see how Theorem 6.3 applies to Zariski dense semi-simple representations of groups. The reader may compare with Theorem 2.1 of [15]. The harmonic map thus constructed is easily seen to be independent of the choice of cusp neighborhoods and is invariant under passing to finite index subgroups of $\Gamma$, i.e. it is canonical in the sense of Definition 6.1. We refer the reader to [40, 41] for uniqueness of finite energy harmonic maps in a considerably more general setting. We summarize this discussion as follows:

**Theorem 6.4.** Let

$$G \notin \{\text{SO}(2,1), \text{SO}(3,1), \text{SU}(2,1)\}$$

be a rank one Lie group and let $X$ be the associated symmetric space. Let $\Gamma < G$ be a non-uniform lattice and $S = X/\Gamma$. Let $\rho: \Gamma \to L$ be a Zariski dense representation from $\Gamma$ to a semi-simple Lie group $L$. Let $Y$ be the symmetric space of $L$. Then there is a canonical $(\Gamma, \rho)$–equivariant harmonic map $h_\rho: X \to Y$.

6.1.2. **Real and complex hyperbolic 2–spaces.** We turn now to two of the exceptional cases: $\{\mathbb{H}^2, \mathbb{C}H^2\}$. Note that when $X$ is real or complex hyperbolic 2–space, then $X$ is a Hermitian symmetric space and $S = X/\Gamma$ has the structure of a quasiprojective variety. In these cases, canonical infinite energy harmonic maps were constructed by Simpson [64] for complex non-compact curves (see also Wolf [68]), and generalized to all quasiprojective varieties by Jost–Zuo [34] and Mochizuki [53, 54]:

**Theorem 6.5.** Let $G \in \{\text{SO}(2,1), \text{SU}(2,1)\}$ and let $X$ be the associated symmetric space. Let $\Gamma < G$ be a non-uniform lattice and $S = X/\Gamma$. Let $\rho: \Gamma \to L$ be a Zariski dense representation from $\Gamma$ to a semi-simple Lie group $L$, and let $Y$ be the symmetric space of $L$. Then there is a canonical $(\Gamma, \rho)$–equivariant harmonic map $h_\rho: X \to Y$.

6.1.3. **Real hyperbolic 3–space.** Finally we turn to the remaining case: $X = \mathbb{H}^3$. In this case, $S = X/\Gamma$ is a non-compact hyperbolic 3–manifold. Let $S^0$ be a compact core as in the discussion preceding Theorem 6.4, bounded by finitely many Euclidean 2–orbifolds $\mathcal{T}_i$ which cut off correspondingly indexed cusps $\mathcal{C}_i$. Fix a (neighborhood of a) cusp $\mathcal{C}$ for notational convenience, and let $\rho: \Gamma \to L$ be a Zariski dense representation from $\Gamma$ to a semi-simple Lie group $L$. As before, we write $Y$ for the symmetric space of $L$. We shall construct a canonical $(\Gamma, \rho)$–equivariant harmonic map $h_\rho: X \to Y$ via a case-by-case analysis of the nature of $\rho|_{\pi_1(\mathcal{C})}$.

In the discussion below, we shall construct initial $\rho|_{\pi_1(\mathcal{C})}$–equivariant model maps from the universal cover $\tilde{C}$ to $Y$. We first fix notation. Let $\mathcal{T} = \partial \mathcal{C}$ denote a horosphere quotient bounding $\mathcal{C}$. Let

$$\mathcal{T}_t = \{x \in \mathcal{C} \mid d(x, \mathcal{T}) = t\}$$

denote equidistant surfaces, i.e. quotients of horospheres at distance $t$ from $\mathcal{T} = \mathcal{T}_0$ inside $\mathcal{C}$. Also, let $\mathcal{C}_t$ denote the closure of the unbounded component of $\mathcal{C} \setminus \mathcal{T}_t$. Write $L = KAN$ for the Iwasawa decomposition of $L$. We elect to use this notation in order to distinguish from $K$ that has been used for a normal subgroup of $\Gamma < G$. For convenience, we pass to a finite index subgroup of $\pi_1(\mathcal{C})$ isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.
Case 1: $\rho|_{\pi_1(C)}$ is discrete faithful and $\rho(\mathbb{Z} \oplus \mathbb{Z})$ consists of semi-simple elements of $L$:

This is the only case where the initial model map is of infinite energy and is the analogue of tame harmonic bundles in the sense of Simpson [64] for non-compact curves (see the SO(2, 1) case discussed above or the model metric near cusps in [34]). It follows by hypothesis that there is an abelian subgroup $A_1$ of rank 2 in $A$ such that $\rho(\mathbb{Z} \oplus \mathbb{Z}) < A_1$. Let $E_1$ be a rank 2 flat stabilized by $A_1$. Fixing a $\rho(\mathbb{Z} \oplus \mathbb{Z})$–equivariant map $f : \tilde{\mathcal{T}} \to A_1$ gives us a simple map in the sense of Lohkamp [47]. This extends to an infinite energy harmonic representative along the cusp $C$ by [47]. Note that for each $t > 0$, we could fix such an initial model map on $C_t$ by demanding that $\tilde{T}_t$ is mapped $\rho(\mathbb{Z} \oplus \mathbb{Z})$–equivariantly to $A$. By [47], each such choice will give us an infinite energy harmonic representative along the cusp $C_t$. The crucial fact that we shall use about all these harmonic representatives is that they lie in a bounded neighborhood of each other (else energy will not be minimized on compact subsets of $C$).

The remaining cases furnish finite energy initial model maps.

Case 2: $\rho|_{\pi_1(C)}$ is faithful and $\rho(\mathbb{Z} \oplus \mathbb{Z})$ lies in the unipotent subgroup $N \subset L$:

We first discuss the case where $\rho|_{\pi_1(C)}$ is discrete. There exists a rank 2 abelian subgroup $N_0$ of $N$ such that $\rho(\mathbb{Z} \oplus \mathbb{Z}) < N_0$. Choose a regular element in $A$ and let $A_0$ be the 1-parameter subgroup generated by it. Then the metric on $A_0N_0$ is Lipschitz equivalent to a metric of pinched negative curvature [46]. Let $\gamma \subset L$ be a parametrized geodesic given by $A_0(o)$ for some base-point $o \in Y$. By stretching the $A_0$–direction by an affine homeomorphism with bi-Lipschitz constant $\kappa \geq 1$ if necessary, we can construct a $\rho|_{\pi_1(C)}$–equivariant map $g_\rho$ from $\tilde{\mathcal{C}}$ to $L$ such that:

1. Each $\mathcal{T}_t$ is mapped to the $N_0$–orbit of $\gamma(kt)$. In particular, $\mathcal{T}_0$ is mapped to the $N_0$–orbit of $o$.
2. In the $t$–direction in $C$, the map $g_\rho$ is an affine stretch with bi-Lipschitz constant at most $\kappa$.
3. On each $\mathcal{T}_t$, $g_\rho$ is 1–Lipschitz in every direction.

It follows that the energy density of $g_\rho$ at every point of $C$ is bounded by $\kappa^2$. Since the volume of $C$ is finite, such an initial model map necessarily has finite energy on $C$.

If $\rho|_{\pi_1(C)}$ is not discrete, then the rank of $N_0$ in the above discussion becomes one instead of two. The rest of the analysis is the same and we have an initial model map of finite energy on $C$.

Case 3: $\rho|_{\pi_1(C)}$ is not faithful:

Passing to a finite-sheeted regular cover of $C$ we may ensure that some primitive element $\gamma$ of $\pi_1(C)$ is in the kernel of $\rho$. Performing Dehn surgery on all the lifts of $C$ closes all the cusps to solid tori and ensures that $\rho$ has a finite energy representative.

Case 4: $\rho|_{\pi_1(C)}$ is indiscrete, faithful and $\rho(\mathbb{Z} \oplus \mathbb{Z})$ lies in a one-parameter subgroup of $L$:

In this case, we can factor the map on $C$ through a map to a hyperbolic solid torus with a cone-metric [33]. Since the cone manifold is compact, there is a finite energy representative.

Case 5: $\rho|_{\pi_1(C)}$ is indiscrete, faithful and $\rho(\mathbb{Z} \oplus \mathbb{Z})$ lies in an abelian subgroup of $L$ generated by two distinct commuting one-parameter subgroups $B_1$ and $B_2$:

At least one of the one-parameter subgroups $B_i$ must lie in the maximal compact $K$. This allows us to again construct a cone-metric on a hyperbolic solid torus so that the map on $C$ factors through it and has a finite energy representative.
Summarizing, we have that for each cusp \( C \) there is either a finite energy initial model map compatible with \( \rho \) in Cases 2-5, or a simple map (in the sense of Lohkamp) in Case 1. Next, recall that harmonic maps in a bounded neighborhood of each other are ‘parallel’, i.e. they differ by at most an isometry moving every point by a fixed distance through harmonic maps (even in the infinite energy case) [27, Section 1] [28, Section 4.3]. Hence fixing initial model maps equivariantly on \( \mathcal{C} \) as above, any two harmonic representatives must be parallel. In particular for semi-simple representations we have uniqueness in the following sense:

**Theorem 6.6.** Let \( G = \text{SO}(3,1) \) and let \( X = \mathbb{H}^3 \) be the associated symmetric space. Let \( \Gamma < G \) be a non-uniform lattice and \( S = X/\Gamma \). Let \( \rho : \Gamma \to L \) be a Zariski dense representation from \( \Gamma \) to a semi-simple Lie group \( L \), and let \( Y \) be the symmetric space of \( L \). Then there is a canonical \((\Gamma,\rho)-\text{equivariant harmonic map} \ h_\rho : X \to Y \).

Note that in constructing \( h_\rho \) in Theorem 6.6, we have, in some cases, been forced to pass to finite index subgroups of \( \Gamma \). In the applications we have in mind, this will not be a problem. Combining Theorems 6.2, 6.4, 6.5, and 6.6 we have the following:

**Theorem 6.7.** Let \( G \) be a rank one Lie group and let \( X \) be the associated symmetric space. Let \( \Gamma < G \) be a lattice and \( S = X/\Gamma \). Let \( \rho : \Gamma \to L \) be a Zariski dense representation from \( \Gamma \) to a semi-simple Lie group \( L \). Let \( Y \) be the symmetric space of \( L \). Then there is a canonical \((\Gamma,\rho)-\text{equivariant harmonic map} \ h_\rho : X \to Y \).

### 6.2 Pseudo-actions and harmonic maps

Let \( Q = \Gamma/K \) and suppose that \( Q \) admits a reductive representation \( \rho_0 \) with Zariski dense image. Let the representation of \( \Gamma \) given by precomposing with the quotient map be called \( \rho \). By Theorem 6.7, there is a canonical harmonic map \( h_\rho : X \to Y \). Let \( \text{Comm}_{pa}(\Gamma,K) \) denote a maximal subgroup of \( \text{Comm}^M(\Gamma,K) \) that pseudo-acts trivially on \( Q \) and which contains \( K^G \) (see Definition 3.14, Theorem 3.11, and Proposition 3.15).

**Lemma 6.8.** The group \( \text{Comm}_{pa}(\Gamma,K) < G \) is discrete.

**Proof.** Since \( K < \text{Comm}_{pa}(\Gamma,K) \), the group \( \text{Comm}_{pa}(\Gamma,K) \) is Zariski dense. Suppose that the group \( \text{Comm}_{pa}(\Gamma,K) \) is not discrete. Then it must be dense. Since \( \rho(\Gamma) \) is also Zariski dense, we may choose a hyperbolic element \( y \in \Gamma \) such that \( \rho(y) \) is a generic semi-simple. In particular, \( \rho(y^n) \) accumulates on a pair of points of the Furstenberg boundary of \( Y \) as \( n \) tends to \( \pm \infty \).

Let \( h_\rho \) be the canonical harmonic map furnished by Theorem 6.7. Since \( \text{Comm}_{pa}(\Gamma,K) \) is dense, there is a dense subset \( V \) of an open neighborhood \( U \) of \( \text{Axis}(y) \) in the space \( (\partial \mathcal{X} \times \partial \mathcal{X}) \setminus \Delta \) of axes in \( \mathcal{X} \), such that each element of \( V \) maps via the harmonic map \( h_\rho \) to a \( \rho(y^n) \)-invariant path for \( N \gg 0 \). Here, we are using the fact that the pseudo-action of \( \text{Comm}_{pa}(\Gamma,K) \) on \( Q \) is trivial. Indeed, for each \( g \in \text{Comm}_{pa}(\Gamma,K) \), we have that

\[ g y^N g^{-1} = y^N \pmod{K} \]

for \( N \gg 0 \). In particular, all of the axes of elements in \( V \) accumulate on the same attracting and repelling points of \( \rho(y^n) \).

The harmonic map \( h_\rho \) is defined on all of \( X \). In particular it is defined on all of the axes in \( U \). Since \( V \) is dense in \( U \), all the axes in \( U \) map to paths accumulating on the same attracting and repelling points of \( \rho(y^n) \). We can cover the space of axes by connected open sets and the previous paragraph forces all axes in \( X \) to map under \( h_\rho \) to paths accumulating on the same attracting and repelling points of \( \rho(y^n) \).

This forces the image of the harmonic map \( h_\rho \) to lie in a flat stabilized by \( \rho(y) \), contradicting Zariski density of \( \rho(Q) \). \( \square \)
Combining Lemma 6.8 with Proposition 3.15, we immediately have:

**Theorem 6.9.** Let $K < \Gamma$ be an infinite normal subgroup of an arithmetic lattice in a rank one Lie group $G$, and let $Q = \Gamma/K$. Further assume that $Q$ admits an infinite image Zariski dense semi-simple representation. Then $\text{Comm}_G(K)$ is discrete.

6.3. Proof of item 1 of Theorem 1.4. We are now in position to establish the first part of our main theorem, which we recall for the reader’s convenience.

**Theorem 6.10.** Let $K < \Gamma$ be an infinite normal subgroup of an arithmetic lattice in a rank one Lie group $G$, and let $Q = \Gamma/K$. Suppose that $Q$ admits an infinite image linear representation. Then $\text{Comm}_G(K)$ is discrete.

**Proof.** Let $\rho : Q \to \text{GL}_n(\mathbb{C})$ be an infinite image linear representation, and let $L$ be the Zariski closure of the image of $\rho$.

Write $U$ for the unipotent radical of $L$. Then, we get $R = L/U$ is a reductive group. Writing $Z$ for the center of $R$, we write $S = R/Z$. We have that $S$ is a semi-simple algebraic group. We write $\rho_U$ and $\rho_S$ for the natural quotient representations of $Q$.

If $S$ has positive dimension, then $Q$ admits an infinite image Zariski dense semi-simple representation and hence Theorem 6.9 gives the desired conclusion. Thus, we may suppose that $S$ is a finite group.

Pulling the trivial group back along the quotient representation $\rho_S : Q \to S$, we obtain a finite index subgroup $\Gamma_0 < \Gamma$ containing $K$ and a quotient group $Q_0 = \Gamma_0/K$ such that the quotient representation $\rho_R : Q_0 \to R$ has image in $Z$. Note that $\Gamma_0$ is again an arithmetic lattice in $G$ and that $\text{Comm}_G(\Gamma_0) = \text{Comm}_G(\Gamma)$.

If $\rho_R(Q_0)$ is infinite then we have that $b_1(Q_0) > 0$ by virtue of it being finitely generated, Theorem 5.10 implies that $\text{Comm}_G(K)$ is discrete.

If $\rho_R(Q_0)$ is finite then by passing to a further finite index subgroup $\Gamma_1 < \Gamma_0$ containing $K$, we obtain a quotient $Q_1 = \Gamma_1/K$ such that $\rho(Q_1) < U < L$. Since $U$ is a unipotent group and since $\Gamma_1$ is finitely generated, it follows that $Q_1$ is finitely generated. It follows again from Theorem 5.10 that $\text{Comm}_G(K)$ is discrete. \qed

7. Generalizations: Proof of items 2 and 3 of Theorem 1.4

In this section we prove items 2 and 3 of Theorem 1.4, using known existence and uniqueness theorems on harmonic maps from the literature.

7.1. Quotients without Kazhdan’s property (T). In this subsection we prove item 2 of Theorem 1.4.

**Theorem 7.1.** Let $\Gamma$ be an arithmetic lattice in $G = \text{SO}(n,1)$ or $G = \text{SU}(n,1)$ such that it is not a low-dimensional exception in the sense of Definition 1.3. Let $K < \Gamma$ be a Zariski dense normal subgroup such that $Q = \Gamma/K$ does not have Kazhdan’s Property (T). Then $\text{Comm}_G(K)$ is discrete.

**Proof.** We refer the reader to [4, Sections 2.1, 2.2] for the relevant background on Kazhdan’s property (T) and Shalom’s remarkable paper [61] (especially Theorem 6.1 there) for refined statements. Since $Q$ does not have Kazhdan’s property (T), there exist:

1. A continuous unbounded affine isometric action of $Q$ on a separable Hilbert space $H$, and hence a continuous unbounded affine isometric action of $\Gamma$ on $H$ (via the initial quotient map from $\Gamma$ to $Q$). Equivalently, there exists a continuous unbounded representation $\Pi_0 : \Gamma \to \text{Aff}(H)$ factoring through $Q$. 


(2) A continuous unitary representation \( \Pi : \Gamma \to \mathcal{U}(\mathcal{H}) \) factoring through \( Q \) such that the associated twisted reduced \( \ell^2 \)-cohomology group is non-trivial: \( \overline{H}^1(\Gamma, \Pi) \neq 0 \). The translation component of the \( \mathcal{H} \)-valued cocycle in Item (1) gives the non-trivial cocycle in \( \overline{H}^1(\Gamma, \Pi) \neq 0 \). In fact \( \Pi \) is the unitary part of \( \Pi_0 \).

As explained in [4], the above can be seen by constructing the flat \( \mathcal{H} \)-bundle associated to the representation \( \Pi \) (via the Borel construction using \( \Pi \)). Then cocycles in \( \overline{H}^1(\Gamma, \Pi) \) are given by the affine translations of \( \mathcal{H} \) and furnish the monodromy of the associated flat \( \mathcal{H} \)-bundle. (When \( \mathcal{H} = \mathbb{R} \) as in Theorem 5.2 we simply have \( H^1(\Gamma, \Pi) = H^1(\Gamma, \mathbb{R}) \).) Choose non-zero \( [\omega] \in \overline{H}^1(\Gamma, \Pi) \). Since we are considering reduced \( \ell^2 \)-cohomology, there is a unique \( \mathcal{H} \)-valued harmonic form \( \omega_{\text{harm}} \) on \( X/\Gamma \) via Hodge theory representing \( [\omega] \), as follows from work of Mok [55] and Korevaar–Schoen [41, Section 2.2]. Equivalently (see Remark 4.9), there exists a \((\Gamma, \rho)\)-equivariant harmonic map \( h_\rho : X \to \mathcal{H} \). We remark that the results of [41] are stated for compact \( S = X/\Gamma \); however, they work equally well for finite energy maps. The discussion in Section 6.1.1 now furnishes existence of a harmonic map. Uniqueness is furnished by the same convexity of distance function argument as in [27, Section 1] (see also the paragraph preceding Theorem 6.6). We furnish some details below.

Let \( g \in \text{Comm}_Q(\mathcal{K}) \). By Theorem 3.11, the \( K^g \)-pseudo-action on \( Q \) is trivial. Hence, so is the induced pseudo-action of any element of \( K^g \) on \( \overline{H}^1(\Gamma, \Pi) \), since \( \Pi : \Gamma \to \mathcal{U}(\mathcal{H}) \) factors through \( Q \). To spell this out along with the uniqueness statement above, we write \( p : X \to X/\Gamma \) for the covering map, and we obtain a \( \Gamma \)-equivariant \( \mathcal{H} \) valued form \( p^* \omega_{\text{harm}} \) on \( X \).

To show that \( p^* \omega_{\text{harm}} \) is invariant under the induced pseudo-action of any element of \( K^g \) we proceed as follows.

**Step 1: The periods of the form \( \omega \) are invariant under \( K^g \):**
This is a reprise of the proof of Lemma 4.8 and we only sketch the computation. Fix a basepoint \( o \in X \) and let \( x \in X \) be equal to \( \gamma(o) \), the image of \( o \) under \( \gamma \) regarded as a deck transformation of \( X \). We let \( \gamma \) be the unique geodesic segment connecting \( o \) to \( x \) (the lift of the corresponding geodesic in \( S = X/\Gamma \)). We can thus define the map

\[
\iota(x) = \int_\gamma p^* \omega_{\text{harm}}.
\]

Since \( p^* \omega_{\text{harm}} \) is \( \Gamma \)-invariant, we have that \( \iota \) furnishes a \( \Gamma \)-equivariant map \( X \to \mathcal{H} \). Since \( \gamma \) is a geodesic segment on \( X \) which descends to a closed geodesic on \( X/\Gamma \) (that is, \( \gamma \) represents a nontrivial element of \( \Gamma \)), then for \( y \in K^g \) we have

\[
\int_{(\gamma^N)y} p^* \omega_{\text{harm}} = \int_{\gamma^N} p^* \omega_{\text{harm}}
\]

for \( N \gg 0 \), since the pseudo-action of \( K^g \) on \( Q \) is trivial. Since

\[
\int_{\gamma^N} p^* \omega_{\text{harm}} = N \cdot \int_\gamma p^* \omega_{\text{harm}},
\]

we have that \( K^g \) preserves the periods of \( p^* \omega_{\text{harm}} \).

**Step 2: Uniqueness of the associated harmonic map \( h_\rho \):**
Let \( X \) be the symmetric space of \( G \). Let \( h_\omega : X \to \mathcal{H} \) be the harmonic map associated to the harmonic form \( \omega_{\text{harm}} \). Note that this is the same as the harmonic map \( h_\rho \) corresponding to the
representation $\Pi_0 : \Gamma \to \text{Aff}(\mathcal{H})$ mentioned in Item (1) above. Invariance of $p^*\omega_{\text{harm}}$ under $K^g$ will follow from uniqueness of $h_p$ as we now argue. Suppose that two harmonic forms $\omega_1$ and $\omega_2$ have the same periods (as in Step 1). Then the associated harmonic maps $h_1$ and $h_2$ will have to agree on $\Gamma(o)$, the $\Gamma$–orbit of the base-point $o$. Hence the quantity

$$||h_1(\gamma(o)) - h_2(\gamma(o))||,$$

i.e. the norm of the difference between these maps, vanishes on $\Gamma(o)$. By convexity of the distance function, the quantity

$$||h_1(x) - h_2(x)||$$

vanishes on all geodesic segments joining $o$ to $x \in \Gamma(o)$. The union of such geodesic segments is dense in $X$. Hence $||h_1(x) - h_2(x)||$ is identically zero. Thus $p^*\omega_{\text{harm}}$ is invariant under $K^g$.

Once we know that $p^*\omega_{\text{harm}}$ is invariant under the induced pseudo-action of any element of $K^g$, the rest of the proof follows the usual scheme (cf. Theorems 5.10 and 6.9, for instance). The group $K^G$ generated by $K^g$ as $g$ ranges over $\text{Comm}_G(K)$ is contained in $\text{Comm}_{pa}(\Gamma, K)$, a maximal subgroup containing $K^G$ of the monoid consisting of $g \in G$ that pseudo-acts trivially on $\mathcal{F}^I(\Gamma, \Pi)$. Note that Steps 1 and 2 above actually show that $\text{Comm}_{pa}(\Gamma, K)$ leaves $p^*\omega_{\text{harm}}$ invariant.

By Proposition 3.15, it remains only to show that $\text{Comm}_{pa}(\Gamma, K)$ is a discrete subgroup of $G$. If $\text{Comm}_{pa}(\Gamma, K)$ is not discrete, it must be dense in $G$ as it contains the Zariski dense subgroup $K$. Thus, we obtain a representation from all of $G$ with image in $\mathcal{U}(\mathcal{H})$ extending $\Pi$ as follows. Indeed, the group $\text{Comm}_{pa}(\Gamma, K)$ preserves the pullback $p^*\omega_{\text{harm}}$, which is an $\mathcal{H}$–valued harmonic form on $X$ via the covering map $p : X \to X/\Gamma$, where the action of $\text{Comm}_{pa}(\Gamma, K)$ on $X$ is by isometries. The continuity of $p^*\omega_{\text{harm}}$, by virtue of it being harmonic, implies that $p^*\omega_{\text{harm}}$ is invariant under the full group of isometries of $X$, since $p^*\omega_{\text{harm}}$ is invariant under the dense subgroup $\text{Comm}_{pa}(\Gamma, K)$. Thus, $p^*\omega_{\text{harm}}$ is invariant under $G$, as claimed. By the equivalence of Items (1) and (2) characterizing (the failure to possess) property (T), we obtain a representation $G \to \mathcal{U}(\mathcal{H})$ extending $\Pi$. Since $\Pi$ is trivial on the Zariski dense subgroup $K$ and since the kernel of $\Pi$ will be a closed subgroup of $G$ by continuity, we have that $\Pi$ must be trivial, a contradiction. Equivalently, there is no representation of all of $G$ into $\text{Aff}(\mathcal{H})$ extending $\Pi_0$. It follows that $\Gamma_0$ is discrete, thus proving the theorem. $\square$

We give an example to which Theorem 7.1 applies. Thompson’s group $T$ is infinite and simple and hence admits no finite dimensional linear representations. However, it admits a $C^\infty$ action on the circle by a result of Ghys–Sergiescu [24], and hence does not have property (T) by a result of Navas [57].

**Remark 7.2.** Theorem 7.1 generalizes the vanishing cuspidal case of Theorem 5.10 as we are using reduced $l^2$ cohomology groups here. However, the non-vanishing cuspidal case of Theorem 5.10 and Theorem 5.7 are not subsumed by it.

### 7.2 $p$–adic lattices and non-positively curved spaces

In this subsection we prove item 3 of Theorem 1.4. In particular $Y$ could be the building corresponding to a $p$–adic semi-simple Lie group $G(\mathbb{Q}_p)$, in which case $\text{Isom}(Y)$ is replaced by $G(\mathbb{Q}_p)$. We first give a geometric definition of semi-simplicity of a representation following Korevaar–Schoen [41] and Labourie [44]:

**Definition 7.3.** Let $Y$ be a proper non-positively curved space. A representation $\rho : \Gamma \to \text{Isom}(Y)$ is said to be semi-simple if

1. Under the induced action of $\Gamma$ on the boundary $\partial Y$, $\rho(\Gamma)$ does not have a global fixed point.
2. There is no proper convex subset of $Y$ preserved by $\rho(\Gamma)$. 
The second condition is a Zariski density condition and takes the place of the condition stated by Labourie in [44] as “sans demies-bandes plates”. For algebraic groups, the second condition allows one to pass from reductive to semi-simple groups; hence the terminology.

The following existence and uniqueness theorem for harmonic representatives now follows from [29, 40, 41, 44, 27].

**Theorem 7.4.** Let $\Gamma$ be a lattice in a rank one Lie group $G$ that is not a low-dimensional exception. Let $X$ be the associated symmetric space. Let $Y$ be a proper non-positively curved space and $\rho : \Gamma \to \text{Isom}(Y)$ be semi-simple. Then there exists a unique $(\Gamma, \rho)$-equivariant harmonic map from $X$ to $Y$.

The existence of a finite energy retraction map from $S = X/\Gamma$ to a compact core was explained in the discussion preceding Theorem 6.4. Existence for general non-positively curved targets is proven in [40, 41] generalizing [29]. Uniqueness is now given by [44] or [27, Section 1]. Given a canonical harmonic representative from Theorem 7.4, the proof of the following Theorem is a straightforward reprise of that of Theorem 6.9 and we omit it.

**Theorem 7.5.** Let $K < \Gamma$ be a Zariski dense normal subgroup of an arithmetic lattice in a rank one Lie group $G$ that is not a low-dimensional exception. Let $Q = \Gamma/K$. Suppose that $Q$ admits a semi-simple representation $\rho : \Gamma \to \text{Isom}(Y)$, where $Y$ is non-positively curved. Then $\text{Comm}_G(K)$ is discrete.

We point out two natural examples that come from other areas of geometry to which Theorem 7.5 applies. All lattices $\Gamma$ in below will be cocompact for convenience of exposition.

**Infinitely generated normal subgroups:** Typically, commutator subgroups (treated in Section 5) are infinitely generated. Another natural class of infinitely generated normal subgroups arises as follows: Let $g \in \Gamma$ be of infinite order. Then for all $n$ sufficiently large, the normal subgroup $K$ generated by $g^n$ in $\Gamma$ is infinitely generated free. More generally if $g$ is a “random word” satisfying small cancellation type conditions, the normal subgroup $K$ generated by $g$ in $\Gamma$ is infinitely generated free [16]. Further, using CAT(-1) small cancellation theory, one can attach a metric 2-cell to $S = X/\Gamma$ to get a new complex $\hat{S}$, such that the universal cover of $\hat{S}$ is non-positively curved. It follows from Theorem 7.5 that $\text{Comm}_G(K)$ is discrete.

**Exotic quotients $Q$:** The Burger-Mozes lattices $Q$ in products of trees [10] are simple groups. In particular, item 1 of Theorem 1.4 does not apply to a normal subgroup $K < \Gamma$ with quotient $Q$. However, they are fundamental groups of non-positively curved spaces. Theorem 7.5 applies. If $Q$ is one of the Thompson’s groups (V, F or T), Farley [21, 22] shows that they act nicely on proper CAT(0) cube complexes. Thus, Theorem 7.5 applies.

**Acknowledgments**

The authors are grateful to Dave Witte Morris and T. N. Venkataramana for helpful discussions and correspondence, in particular for telling us about Proposition 2.3, and to David Fisher for helpful discussions. We thank François Labourie for helpful correspondence and references on harmonic maps. We thank Dani Wise for his lectures on coherence at IMPAN, Warsaw, and for references on $L^2$-betti numbers [17]. Finally we thank Yehuda Shalom for his interest and perceptive comments.
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