OPERAD BIMODULES, AND COMPOSITION PRODUCTS
ON ANDRÉ–QUILLEN FILTRATIONS OF ALGEBRAS

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Abstract. If $\mathcal{O}$ is a reduced operad in a symmetric monoidal category of spectra ($S$–modules), an $\mathcal{O}$–algebra $I$ can be viewed as analogous to the augmentation ideal of an augmented algebra. Implicit in the literature on Topological André–Quillen homology is that such an $I$ admits a canonical (and homotopically meaningful) decreasing $\mathcal{O}$–algebra filtration $I \leftarrow I^1 \leftarrow I^2 \leftarrow I^3 \leftarrow \ldots$ satisfying various nice properties analogous to powers of an ideal in a ring.

In this paper, we are explicit about these constructions. With $R$ a commutative $S$–algebra, we use a bar construction as a derived version of functors of the form $I \mapsto M \circ_{\mathcal{O}} I$, where $M$ is an $\mathcal{O}$–bimodule, and $I$ is an $\mathcal{O}$–algebra in $R$–modules. We then note that the composition structure of the operad induces products $(I^i)^j \rightarrow I^j$, fitting nicely with previously studied structure. As a formal consequence, an $\mathcal{O}$–algebra map $I \rightarrow J^d$ induces compatible maps $I^n \rightarrow J^{dn}$ for all $n$. This is an essential tool in the first author’s study of Hurewicz maps for infinite loop spaces, and its utility is illustrated here with a lifting theorem.

1. Introduction

Let $S$–mod be the category of symmetric spectra [HSS], one of the standard symmetric monoidal models for the category of spectra. Let $S$ denote the sphere spectrum, and let $\mathcal{O}$ be a reduced operad in $S$–mod. If $R$ is a commutative $S$–algebra, we let $\text{Alg}_{\mathcal{O}}(R)$ denote the category of $\mathcal{O}$–algebras in $R$–modules.

The starting point of this paper is the observation that, if $M$ is a reduced $\mathcal{O}$–bimodule, and $I \in \text{Alg}_{\mathcal{O}}(R)$, then $M \circ_{\mathcal{O}} I$ is again in $\text{Alg}_{\mathcal{O}}(R)$, and that many interesting constructions on $\mathcal{O}$–algebras are derived versions of functors of $I$ of this form.

Our first goal here is to present the basic properties of a suitable derived version of this construction, the bar construction $B(M, \mathcal{O}, I)$, noting how structure on the category of $\mathcal{O}$–bimodules is reflected in the category of endofunctors of $\mathcal{O}$–algebras. Perhaps the least familiar of these is that a levelwise homotopy cofibration sequence in the bimodule variable $M$ induces a homotopy fibration sequence in $\text{Alg}_{\mathcal{O}}(R)$: see Theorem 2.10(b).

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We then take advantage of the observation that a pairing of bimodules
\[ L \circ_\mathcal{O} M \to N \]
will induce a natural transformation of functors of \( I \)
\[ L \circ_\mathcal{O} (M \circ_\mathcal{O} I) \to N \circ_\mathcal{O} I, \]
and similarly on our derived model.

This leads to some new structure on some much studied constructions on \( \mathcal{O} \)-algebras. An \( \mathcal{O} \)-algebra \( I \) can be viewed as similar to the augmentation ideal in an augmented ring. Implicit in the literature, e.g., [HH], is that \( I \in \text{Alg}_{\mathcal{O}}(R) \) admits a homotopically meaningful natural 'augmentation ideal filtration':
\[ I \leftarrow I^1 \leftarrow I^2 \leftarrow I^3 \leftarrow \ldots \]
with \( I^n/I^{n+1} \), determined by \( \mathcal{O}(n) \) and the topological André-Quillen \( R \)-module \( TQ(I) \).\(^1\) Furthermore, if \( R \) and \( \mathcal{O} \) are connective and \( I \) is \((c-1)\)-connected, then \( I^n \) will be \((nc-1)\)-connected.

Our constructions, applied to well chosen pairings among \( \mathcal{O} \)-bimodules, then yield pairings
\[ (I^n)_m \to I^{mn}. \]

This seems to be fundamental structure which has not previously appeared in the literature. Here is a consequence illustrating its utility:

**Theorem 1.1.** Let \( f : I \to J \) be a map in the homotopy category \( \text{hoAlg}_\mathcal{O}(R) \). If \( f \) factors as \( f = f_s \circ \cdots \circ f_1 \) such that \( TQ(f_i) \) is null for all \( i \), then there is a lifting in \( \text{hoAlg}_\mathcal{O}(R) \):
\[ \begin{array}{c}
I \\
\downarrow f \\
\downarrow J
\end{array} \]
\[ \begin{array}{c}
J^2 \\
\downarrow J^2
\end{array} \]
\[ \begin{array}{c}
J \\
\uparrow \tilde{f}
\end{array} \]

(We restate this, with slightly different notation, as Theorem 3.10.)

Further applications in this spirit can be seen in work by the first author on Hurewicz maps of infinite loopspaces [K3], the project whose needs motivated this paper. Also critical in [K3], is that we verify that there are sensible ‘change-of-rings’ formulae for our constructions as \( R \) varies.

The paper is organized as follows.

In §2, we first introduce the setting in which we wish to work. This includes a well chosen, and slightly delicate, model structure on \( \text{Alg}_\mathcal{O}(R) \), which piggybacks off of the ‘positive’ model structure on \( S \)-mod first exploited in [S], and is in the spirit of [H1]. We then state the basic homotopy properties of our derived version of \( M \circ_\mathcal{O} I \). To summarize, what we find most compelling is that on one hand, our constructions connect nicely to \( TQ(I) \), and on the other, they are well suited to iteration using the monoidal properties of \( \circ \).

\(^1\) \( TQ(I) \) can be informally viewed as ‘\( I/I^2 \)’, and its study goes back to [B].
In §3, we apply the result of the previous section to the augmentation ideal filtration of an \( \mathcal{O} \)-algebra, and deduce lifting results as above. The deeper proofs from §2 are deferred to §4, which itself is supported by Appendix A. Much of the technical work consists of generalizing results and arguments from [P2] from \( S \)-mod to \( R \)-mod for a general \( R \).

2. GENERAL RESULTS ABOUT DERIVED COMPOSITION PRODUCTS

2.1. Our categories of modules and algebras. In this paper, the category of \( S \)-modules will mean the category of symmetric spectra as defined in [HSS]: here we recall that \( X \in S\text{-mod} \) consists of a sequence \( X_0, X_1, X_2, \ldots \) of simplicial sets equipped with extra structure.

With the smash product as product and sphere spectrum \( S \) as unit, \( S\text{-mod} \) is a closed symmetric monoidal category. There is a notion of weak equivalence, and various model structures on \( S\text{-mod} \) compatible with these, such that the resulting quotient category models the standard stable homotopy category.

Recall that a symmetric sequence in \( S\text{-mod} \) then consists of a sequence \( X(0), X(1), X(2), \ldots \) where \( X(n) \) is a symmetric spectrum equipped with an action of the \( n \)th symmetric group \( \Sigma_n \).

The category of such symmetric sequences in \( S\text{-mod} \), \( \text{Sym}(S) \), admits a composition product \( \circ \) defined by

\[
(X \circ Y)(s) = \bigvee_r X(r) \wedge_{\Sigma_r} \left( \bigvee_{\phi: s \to r} Y(s_1(\phi)) \wedge \ldots \wedge Y(s_r(\phi)) \right),
\]

where \( s = \{1, \ldots, s\} \) and \( s_k(\phi) \) is the cardinality of \( \phi^{-1}(k) \). With this product, \( \text{Sym}(S), \circ, S(1) \) is monoidal, where \( S(1) = (*, S, *, S, \ldots) \).

An operad \( \mathcal{O} \) is then a monoid in this category, and one makes sense of left \( \mathcal{O} \)-modules, right \( \mathcal{O} \)-modules, and \( \mathcal{O} \)-bimodules in the usual way. Furthermore, if \( X \) is a right \( \mathcal{O} \)-module, and \( Y \) is a left \( \mathcal{O} \)-module, the symmetric sequence \( X \circ_{\mathcal{O}} Y \) can be defined in the usual way, as the coequalizer in \( \text{Sym}(S) \) of the two evident maps

\[ X \circ \mathcal{O} \circ Y \rightrightarrows X \circ Y. \]

Extra structure on \( X \) or \( Y \) then can induce evident extra structure on \( X \circ_{\mathcal{O}} Y \).

For the purposes of this paper, it is natural to require that our operads \( \mathcal{O} \) and bimodules \( M \) be reduced: \( \mathcal{O}(0) = * = M(0) \). By contrast, an \( \mathcal{O} \)-algebra is a left \( \mathcal{O} \)-module \( I \) concentrated in level 0: \( I(n) = * \) for all \( n > 0 \).

If \( R \) is a commutative \( S \)-algebra, these definitions and constructions extend to the category of \( R \)-modules. Furthermore, one can mix and match: for example, if \( X \) is a symmetric sequence in \( S\text{-mod} \) and \( Y \) is a symmetric
sequence in $R$-mod, $X \circ Y$ will be the symmetric sequence in $R$-mod with

$$(X \circ Y)(s) = \bigvee_r X(r) \wedge \Sigma_r \left( \bigvee_{\phi: r \to r} Y(s_1(\phi)) \wedge_R \ldots \wedge_R Y(s_r(\phi)) \right).$$

We denote by $\text{Sym}(R)$ the category of symmetric sequences in $R$-mod, $\text{Alg}_O(R)$ the category of $O$-algebras in $R$-mod and $\text{Mod}_O(R)$ the category of left $O$-modules in $\text{Sym}(R)$.

2.2. Model structures. We specify model structures on the various categories just described.

We accept as given the $S$–model structure on symmetric spectra (called $S$–modules in this paper) as defined in [HSS, Defn.5.3.6] and [S, Thm.2.4]. This structure is monoidal with respect to the smash product [HSS, Cor.5.3.8].

We then give $\text{Sym}(S)$ its associated injective model structure: weak equivalences and cofibrations are those morphisms which are levelwise weak equivalences and cofibrations in $S$-mod. That this structure exists was checked in [P2].

As in [MMSS, §15], [S], [HH], and [P2], we need ‘positive’ variants of these model structures. Weak equivalences will be as before, but there are fewer cofibrations: for $X \to Y$ in $S$-mod to be a positive cofibration, we now insist that $X_0 \to Y_0$ also be an isomorphism, and for $M \to N$ in $\text{Sym}(S)$ to be a positive cofibration, we now insist that $M(0)_0 \to N(0)_0$ also be an isomorphism\(^3\). It is worth noting that if $M \in \text{Sym}(S)$ is reduced, then it is positive cofibrant exactly when each $M(n)$ is cofibrant, when viewed in $S$-mod.

Given a commutative $S$–algebra $R$, the positive $R$–model structure on $R$–modules is then defined to be the projective structure induced from that on $S$-mod with its positive structure: weak equivalences and fibrations in $R$-mod are the maps which are weak equivalences and positive fibrations in $S$-mod. Similarly, we define the positive structure on $\text{Sym}(R)$, the category of symmetric sequences in $R$-mod, to be the projective structure induced from that on $\text{Sym}(S)$ with its positive structure: weak equivalences and fibrations in $\text{Sym}(R)$ are the maps which are weak equivalences and positive fibrations in $\text{Sym}(S)$.

The following theorem is proved in [P2] (see also [H1]), with special cases going back to [S].

**Theorem 2.1.** $\text{Alg}_O(R)$ has a projective model structure induced from the positive structure on $R$-mod: $f : I \to J$ is a weak equivalence if it is one in $R$-mod (and thus in $S$-mod), and a fibration if it is a positive fibration in $R$-mod (and thus in $S$-mod). Similarly, $\text{Mod}_O(R)$ has a projective model

\(^2\)This structure is different from the associated projective structure used in [H1, H2, HH].

\(^3\)On $\text{Sym}(S)$, this agrees with [P2] but is different from [HH], where it is required that $M(n)_0 \to N(n)_0$ be an isomorphism for all $n$. 
structure induced from the positive structure on $\text{Sym}(R)$: $f : M \to N$ is a weak equivalence if it is one in $\text{Sym}(R)$ (and thus in $\text{Sym}(S)$), and a fibration if it is a positive fibration in $\text{Sym}(R)$ (and thus in $\text{Sym}(S)$).

The next lemma says that the model structure on $\text{Alg}_O(R)$ is really the same as the model structure on $\text{Mod}_O(R)$, restricted to the subcategory of modules concentrated in degree 0.

**Lemma 2.2.** If $I \to J$ is a cofibration in $\text{Alg}_O(R)$, then it is a cofibration in $\text{Mod}_O(R)$, when $I$ and $J$ are regarded as objects in $\text{Sym}(R)$ concentrated in level 0.

**Proof.** The inclusion $\text{Alg}_O(R) \hookrightarrow \text{Mod}_O(R)$ has right adjoint given by $M \mapsto M(0)$. This is a Quillen pair, as it is easily checked that this right adjoint preserves weak equivalences and fibrations.

2.3. **Cofibrancy assumption on $O$ and first consequences.** Unless stated otherwise, we make the following standing cofibrancy assumption about our operad $O$.

**Assumption 2.3.** The map $S(1) \to O$ is assumed to be a positive cofibration in $\text{Sym}(S)$.

As $O(0) = \ast$ has been assumed earlier, equivalently this means that, in $S\text{-mod}$, $S \to O(1)$ is a cofibration, and $O(n)$ is cofibrant for all $n$.

**Notation 2.4.** Let $\text{Alg}_O(R)^c$ be the full subcategory of $\text{Alg}_O(R)$ consisting of $O$–algebras in $R\text{-mod}$ which are cofibrant when just viewed as $R$–modules.

A key advantage of our particular model structure on $\text{Alg}_O(R)$ is that the following property holds.

**Proposition 2.5.** The forgetful functor $\text{Alg}_O(R) \to R\text{-mod}$ preserves cofibrations between cofibrant objects. In particular, if $I$ is cofibrant in $\text{Alg}_O(R)$, then $I \in \text{Alg}_O(R)^c$.

When $R = S$ this is [P2, Theorem 1.6]. We defer the proof of the general case to §4.

It follows that a functorial cofibrant replacement functor

$$c^R : \text{Alg}_O(R) \to \text{Alg}_O(R)$$

takes values in $\text{Alg}_O(R)^c$.

More elementary, but also useful is that $\text{Alg}_O(R)^c$ is well behaved under change of rings.

**Lemma 2.6.** Let $R \to R'$ be a map of commutative $S$–algebras. Then

$$R' \wedge_R - : \text{Alg}_O(R) \to \text{Alg}_O(R')$$

restricts to a functor

$$R' \wedge_R - : \text{Alg}_O(R)^c \to \text{Alg}_O(R')^c$$

which preserves weak equivalences.
Proof. This is immediate since $R' \wedge_R$ is left adjoint to a forgeful functor that is easily seen to be right Quillen. □

2.4. General properties of the bar construction. We will make much use of the bar construction. Given an $\mathcal{O}$–bimodule $M$ and $I \in \text{Alg}_\mathcal{O}(R)$, $B(M, \mathcal{O}, I) \in \text{Alg}_\mathcal{O}(R)$ is defined as the geometric realization of the simplicial object $B_\bullet(M, \mathcal{O}, I)$ in $R$-mod defined by

$$B_n(M, \mathcal{O}, I) = M \odot \mathcal{O} \odot \cdots \odot \mathcal{O} \odot I.$$ 

Similarly if $M$ and $N$ are $\mathcal{O}$–bimodules, then $B(M, \mathcal{O}, N)$ is again an $\mathcal{O}$-bimodule.

The theme of the next set of results is that this construction is well-behaved when the $\mathcal{O}$–bimodules are positive cofibrant in $\text{Sym}(\mathcal{S})$, and $I \in \text{Alg}_\mathcal{O}(R)$ is cofibrant in $R$-mod. (We recall that a reduced $M \in \text{Sym}(\mathcal{S})$ is positively cofibrant exactly when it is levelwise cofibrant.)

Proposition 2.7. Let $M$ and $N$ be levelwise cofibrant $\mathcal{O}$–bimodules. Then $B(M, \mathcal{O}, N)$ is again levelwise cofibrant. Similarly, if $M$ is levelwise cofibrant and $I \in \text{Alg}_\mathcal{O}(R)^c$, then $B(M, \mathcal{O}, I) \in \text{Alg}_\mathcal{O}(R)^c$.

The first statement is immediately implied by [P2, Theorem 1.7] which says that $B_\bullet(M, \mathcal{O}, N)$ is Reedy cofibrant in the category of simplicial objects in $\text{Sym}(\mathcal{S})$. We defer the proof of the second statement for general $R$ to §4.

We also record the following, which shows that the bar construction can be usefully used as a derived circle product.

Proposition 2.8. Let $M$ be a levelwise cofibrant right $\mathcal{O}$–module. If $I$ is cofibrant in $\text{Alg}_\mathcal{O}(R)$, the natural map $B(M, \mathcal{O}, I) \to M \circ \mathcal{O} I$ is a weak equivalence. Similarly if $N$ is cofibrant in $\text{Mod}_\mathcal{O}(\mathcal{S})$, then $B(M, \mathcal{O}, N) \to M \circ \mathcal{O} N$ is a weak equivalence.

This will also be proved in §4.

To emphasize the functors defined by levelwise cofibrant bimodules, we change notation.

Definition 2.9. If $M$ is a levelwise cofibrant $\mathcal{O}$–bimodule, define

$$F_M^R : \text{Alg}_\mathcal{O}(R)^c \to \text{Alg}_\mathcal{O}(R)^c$$

by the formula $F_M^R(I) = B(M, \mathcal{O}, I)$.

Theorem 2.10. The $F_M^R$ construction satisfies the following properties.

(a) $(M, I) \mapsto F_M^R(I)$ takes weak equivalences in either the $M$ or $I$ variable to weak equivalences in $\text{Alg}_\mathcal{O}(R)$.

(b) A levelwise homotopy cofibration sequence of levelwise cofibrant $\mathcal{O}$–bimodules

$$L \to M \to N$$

induces a homotopy fibration sequence in $\text{Alg}_\mathcal{O}(R)$

$$F_L^R(I) \to F_M^R(I) \to F_N^R(I).$$
There is a natural isomorphism of functors:

\[ F_R^M \circ F_R^N \simeq F_R^{B(M,O,N)}. \]

Let \( R \to R' \) be a map of commutative \( S \)-algebras. There is a natural isomorphism in \( \text{Alg}_O(R') \):

\[ F_{R'}^M(R' \wedge_R I) \simeq R' \wedge_R F_{M}^{R}(I). \]

Parts (a) and (b) will be proved in §4. By contrast, parts (c) and (d) are straightforward. Part (c) follows from the natural isomorphism

\[ B(M,O,B(N,O,I)) \simeq B(B(M,O,N),O,I), \]

while part (d) follows from the natural isomorphism

\[ R' \wedge_R B(M,O,I) \simeq B(M,O,R' \wedge_R I). \]

Remark 2.11. As there is a natural map \( B(M,O,N) \to M \circ_O N \), it follows that a bimodule pairing

\[ \mu : M \circ_O N \to L \]

induces a natural transformation

\[ \mu : F_R^M \circ F_R^N \to F_R^L \]

defined as the composite

\[ F_R^M \circ F_R^N \simeq F_R^{B(M,O,N)} \to F_R^{M \circ_O N} \to F_R^L. \]

See §3 for examples of this.

2.5. \( O \)-bimodules with one term and André–Quillen homology. In this subsection, we consider our constructions when \( M \) is concentrated in just one level, i.e., there exists an \( n \) such that \( M(m) = * \) for all \( m \neq n \). We show that then \( F_R^M(I) \) is determined by the Topological André–Quillen homology of \( I \).

We first need to define this last last term in our context. The \( S \)-module \( O(1) \) will be an associative \( S \)-algebra, and can be viewed as an operad concentrated in level 1. From this point of view, the evident maps \( O(1) \to O \) and \( O \to O(1) \) are both maps of operads, and the second of these gives \( O(1) \) the structure of an \( O \)-bimodule concentrated in level 1.

Let \( R O(1)\text{-mod} \) be the category of \( R \wedge O(1) \)-modules. It is illuminating to note that this category is also \( \text{Alg}_{O(1)}(R) \), when one views \( O(1) \) as an operad. The map \( O \to O(1) \) induces a functor

\[ z : R O(1)\text{-mod} \to \text{Alg}_O(R) \]

with left adjoint

\[ Q = O(1) \circ_O - : \text{Alg}_O(R) \to R O(1)\text{-mod} \]

making the pair of functors into a Quillen pair.

Definition 2.12. Define \( TQ : \text{Alg}_O(R)^c \to R O(1)\text{-mod} \) by the formula

\[ TQ(I) = B(O(1),O,I). \]
The next proposition is a special case of Proposition 2.8.

**Proposition 2.13.** If $I$ is cofibrant in $\text{Alg}_O(R)$, the natural map $TQ(I) \rightarrow Q(I)$ is an equivalence.

As $TQ$ is thus equivalent to the left derived functor of the left Quillen functor $Q$, one has the next two consequences.

To state the first, we let $[I,J]_{\text{Alg}}$ denote morphisms between $I$ and $J$ in the homotopy category of $\text{Alg}_O(R)$, and we similarly let $[M,N]_{\text{Mod}}$ denote morphisms between $M$ and $N$ in the homotopy category of $RO(1)$-mod.

**Corollary 2.14.** There is an adjunction in the associated homotopy categories:

$$[TQ(I),N]_{\text{Mod}} \simeq [I,z(N)]_{\text{Alg}}.$$ 

**Corollary 2.15.** If $I \rightarrow J \rightarrow K$ is a homotopy cofibration sequence in $\text{Alg}_O(R)$, then

$$TQ(I) \rightarrow TQ(J) \rightarrow TQ(K)$$

is a homotopy cofibration sequence in $RO(1)$-mod.

The next result is a particular instance of Theorem 2.10(d).

**Proposition 2.16.** Let $R \rightarrow R'$ be a map of commutative $S$-algebras. There is a natural isomorphism

$$TQ(R' \wedge_R I) \simeq R' \wedge_R TQ(I).$$

(The first ‘$TQ$’ here is, of course, with respect to the $S$-algebra $R'$.)

Now we note that, if $M$ is an $O$-bimodule, then each $M(n)$ will be a left $O(1)$-module and a right $\Sigma_n \wr O(1)$-module, where $\Sigma_n \wr O(1)$ is the associative algebra with underlying $S$-module $\coprod_{\sigma \in \Sigma_n} O(1)^{\wedge n}$, and evident ‘twisted’ multiplication.

Conversely, given ‘$M(n)$’, a left $O(1)$-module that is also a right $\Sigma_n \wr O(1)$-module, we can view the symmetric sequence

$$z(M(n)) = (\ast, \ldots, \ast, M(n), \ast, \ldots)$$

as a bimodule over either $O(1)$, viewed as an operad, or $O$.

**Theorem 2.17.** If $M(n)$ is also a cofibrant $S$-module, for $I \in \text{Alg}_O(R)^c$, there are natural equivalences:

$$F^R_{z(M(n))}(I) \simeq B(z(M(n)),O(1),TQ(I)) \simeq z(M(n) \wedge_{\Sigma_n \wr O(1)} TQ(I)^{\wedge n}).$$

**Proof.** After unraveling definitions one sees that both the first and last terms are isomorphic to $B(z(M(n)),O(1),I)$, and hence certainly weakly equivalent.

Further, the first map can be identified with

$$F^R_{z(M)}(I) \leftarrow F^R_{B(z(M),O(1),O(1))}(I),$$

which is a w.e. due to part (a) of Theorem 2.10 together with the standard fact that $B(z(M),O(1),O(1)) \rightarrow z(M)$ is a w.e..
Corollary 2.18. Let $f : I \to J$ be a morphism in $\text{Alg}_O(R)^c$. With $M(n)$ as in the theorem, if $TQ(f)$ is a weak equivalences, so is $F^R_{z(M(n))}(f)$.

2.6. The Goodwillie tower of $F^R_M$. The second author has studied Goodwillie calculus on the category $\text{Alg}_O(R)$ [P1]. Here we sketch how our results above lead to an understanding of the Goodwillie tower of the functor $F^R_M$.

Given a levelwise cofibrant $O$–bimodule $M$, let $M^\leq n$ denote the $O$–bimodule with

$$M^\leq n(k) = \begin{cases} M(k) & \text{if } k \leq n \\ * & \text{if } k > n. \end{cases}$$

Definition 2.19. Let $P_nF^R_M = F^R_{M^\leq n} : \text{Alg}_O(R)^c \to \text{Alg}_O(R)^c$.

Theorem 2.20. The Goodwillie tower of the functor $F^R_M$ identifies with

$$P_1F^R_M \leftarrow P_2F^R_M \leftarrow P_3F^R_M \leftarrow \ldots,$$

and its $n$th derivative $\partial_nF^R_M$ identifies with $M(n)$.

Sketch proof. The sequence of $O$-bimodules

$$z(M(n)) \to M^\leq n \to M^\leq (n-1)$$

satisfies the hypothesis of Theorem 2.10(b). Thus the homotopy fiber of the map

$$P_nF^R_M(I) \to P_{n-1}F^R_M(I)$$

identifies as $F^R_{z(M(n))}(I)$, which Theorem 2.17 tells us is

$$z(M(n)) \wedge_{\Sigma_nO(1)} TQ(I)^{\wedge R^n}.$$ 

This is a homogeneous $n$–excisive functor: note that Corollary 2.15 first tells us that $TQ$ is a homogeneous linear functor. (See [P1, Theorem 3.2] for more detail.)

It follows that $P_nF^R_M$ is $n$–excisive. With a bit more care, one can now check that the natural transformation $F^R_M \to P_nF^R_M$ identifies with the map from $F^R_M$ to its $n$–excisive quotient: the proof of [P1, Theorem 4.3] generalizes immediately to our setting. □

Under connectivity hypotheses, one gets very concrete convergence estimates. Say that $X \in \text{Sym}(S)$ is connective if each $X(n) \in S$-mod is connective, i.e. $-1$–connected.

Proposition 2.21. If $R$, $M$, and $O$ are connective, and $I$ is $(c-1)$–connected, then the map $F^R_M(I) \to P_nF^R_M(I)$ is $(n+1)c$–connected.

Proof. We need to show that the homotopy fiber is $((n+1)c-1)$–connected. By Theorem 2.10(b), this homotopy fiber identifies with $B(M^\geq n, O, I)$ where

$$M^\geq n(k) = \begin{cases} M(k) & \text{if } k > n \\ * & \text{if } k \leq n. \end{cases}$$
This fiber then is the homotopy colimit (in $R$–modules) of a diagram of $R$–modules of the form

$$M(r) \wedge \mathcal{O}(s_1) \wedge \ldots \wedge \mathcal{O}(s_k) \wedge I^\wedge t,$$

with $t \geq r > n$. In particular, it is a homotopy colimit of a diagram of $((n+1)c-1)$–connected $R$–modules, and so is itself $((n+1)c-1)$–connected. \hfill \square

These results also show the following, when combined with Corollary 2.18.

**Corollary 2.22.** Let $f : I \to J$ be a morphism in $\text{Alg}_{\mathcal{O}}(R)^c$. If $TQ(f)$ is a weak equivalence, so is $P_nF^R_{\mathcal{O}}(f)$ for any $n$ and any levelwise cofibrant $\mathcal{O}$–bimodule $M$. Furthermore, if $R$, $M$, and $\mathcal{O}$ are connective, and $I$ and $J$ are 0–connected, then $f$ is itself will be a weak equivalence.

### 3. Application to the augmentation ideal filtration

In our constructions, when the $\mathcal{O}$–bimodule is $\mathcal{O}$ itself, the resulting functor $I \mapsto F^R_{\mathcal{O}}(I) = B(\mathcal{O}, \mathcal{O}, I)$ is naturally weakly equivalent to the identity. In this section we study structure on the ‘augmentation ideal’ filtration of $I$ arising from using the levelwise bimodule filtration of $\mathcal{O}$ in conjunction with the operad structure $\mathcal{O} \circ \mathcal{O} \to \mathcal{O}$.

#### 3.1. Construction and basic properties of the filtration.

**Definitions 3.1.** Let $1 \leq i < m \leq \infty$.

(a) Let $\mathcal{O}^m_i$ denote the $\mathcal{O}$–bimodule with $\mathcal{O}^m_i(k) = \begin{cases} \mathcal{O}(k) & \text{if } i \leq k < m \\ * & \text{otherwise} \end{cases}$.

(b) For $I \in \text{Alg}_{\mathcal{O}}(R)^c$, let $I^m_i = F^R_{\mathcal{O}^m_i}(I) = B(\mathcal{O}^m_i, \mathcal{O}, I)$.

Note that there is a natural weak equivalence $I^1_\infty \to I$. We sometimes write $I^i$ for $I^i_\infty$, and readers are encouraged to view $I^m_i$ as $\mathcal{O}_i/I^m_i$.

For $j \leq i$ and $n \leq m$, it is not hard to see that the evident map

$$\mathcal{O}^m_i \to \mathcal{O}^m_n$$

is a map of $\mathcal{O}$–bimodules, and thus induces a natural maps

$$I^m_i \to I^m_n$$

for all $I \in \text{Alg}_{\mathcal{O}}(R)^c$.

Special cases of these are illustrated in the next examples.

**Example 3.2.** $I \in \text{Alg}_{\mathcal{O}}(R)^c$ has a natural ‘augmentation ideal’ filtration

$$I \leftarrow I^1 \leftarrow I^2 \leftarrow I^3 \leftarrow \ldots.$$

**Example 3.3.** $I^1_n = P_n-1F^R_{\mathcal{O}}(I)$ in the notation of the last section, so the tower

$$I^1_2 \leftarrow I^1_3 \leftarrow I^1_4 \leftarrow \ldots$$

identifies with the Goodwillie tower of the identity functor on $\text{Alg}_{\mathcal{O}}(R)$. 
These examples are related: the filtration of the first example appears as the homotopy fibers of the maps from $I$ to the tower in the second example. More precisely, there are homotopy fiber sequences

$$I^n \to I^1 \to I^1_n.$$  

This is a special case of property (b) in the next theorem.

**Theorem 3.4.** The functors $I \mapsto I^i_n$ satisfy the following properties.

(a) They preserve weak equivalences in the variable $I \in \text{Alg}_{\mathcal{O}}(R)$.  
(b) For $k < l < m$, the sequence $I^n_m \to I^k_m \to I^k_l$ is a homotopy fiber sequence. 
(c) $I^1_2 \simeq z(TQ(I))$. More generally, $I^k_{k+1} \simeq z(\mathcal{O}(k) \wedge_{\Sigma_k} \mathcal{O}(1) TQ(I)^{\wedge Rk})$. 
(d) Let $R \to R'$ be a map of commutative $S$–algebras. There is a natural isomorphism $R' \wedge_R I^i_n \simeq (R' \wedge_R I)^i_n$.

All of these properties follow immediately from the more general results of §2. For example, part (b) follows from Theorem 2.10(b) applied to the sequence of $\mathcal{O}$–bimodules

$$\mathcal{O}_m^r \to \mathcal{O}_k^m \to \mathcal{O}_k^l.$$  

Our connectivity estimates of §2.6 give the following.

**Proposition 3.5.** Suppose $R$ and $\mathcal{O}$ are connective. If $I$ is $(c-1)$–connected, then $I^n$ is $(nc-1)$–connected.

3.2. Composition properties of the filtration. Now we look at composition structure. It is not hard to see that the operad composition

$$\mu : \mathcal{O} \circ \mathcal{O} \to \mathcal{O}$$

induces maps of $\mathcal{O}$–bimodules

$$\mu : \mathcal{O}_i^\infty \circ \mathcal{O}_j^\infty \to \mathcal{O}_{ij}^\infty.$$ 

These pairings in turn define natural maps

$$\mu : (I^i)^j \to I^{ij}$$

for all $I \in \text{Alg}_{\mathcal{O}}(R)$.  

With a little more care, one can check the following.

**Lemma 3.6.** $\mu : \mathcal{O} \circ \mathcal{O} \to \mathcal{O}$ induces maps of $\mathcal{O}$–bimodules

$$\mu : \mathcal{O}_m^r \circ \mathcal{O}_j^n \to \mathcal{O}^{\min(ij+(n-j),mj)}_{ij},$$ and thus induces natural maps

$$\mu : (I^n)^i \to I^{ij}_{\min(ij+(n-j),mj)},$$
Proof. We first check that if \( N = \min(ij + (n - j), mj) \), then the dotted arrow exists in the diagram

\[
\begin{array}{ccc}
\mathcal{O}_i^\infty \circ \mathcal{O}_j^\infty & \longrightarrow & \mathcal{O}_{ij}^\infty \\
\downarrow & & \downarrow \\
\mathcal{O}_i^n \circ \mathcal{O}_j^n & \longrightarrow & \mathcal{O}_{ij}^N.
\end{array}
\]

Now \((\mathcal{O}_i^\infty \circ \mathcal{O}_j^\infty)(s)\) equals the wedge of \(S\)-modules of the form \(\mathcal{O}(r) \wedge \mathcal{O}(s_1) \wedge \ldots \wedge \mathcal{O}(s_r)\) such that \(s = s_1 + \cdots + s_r, i \leq r, \) and \(j \leq s_k\) for all \(k\). (All such modules occur, some with multiplicities greater than 1.)

Such a wedge summand maps to * under the quotient \(\mathcal{O}_i^\infty \circ \mathcal{O}_j^\infty \to \mathcal{O}_i^m \circ \mathcal{O}_j^n\) if either \(r \geq m\) or \(s_k \geq n\) for at least one \(k\). In the first case, it follows that \(s \geq mj\). In the second case, it follows that \(s \geq (r - 1)j + n \geq (i - 1)j + n = ij + (n - j)\). We conclude that if \(N\) is less than or equal to both \(mj\) and \(ij + (n - j)\), then the dotted arrow in the diagram above exists.

The bimodule map \(\mathcal{O}_i^m \circ \mathcal{O}_j^n \to \mathcal{O}_{ij}^{\min(ij + (n - j), mj)}\) then induces an \(\mathcal{O}\)-bimodule map \(\mathcal{O}_i^m \circ \mathcal{O}_j^n \to \mathcal{O}_{ij}^{\min(ij + (n - j), mj)}\). This follows formally from the fact that each of the maps \(\mathcal{O} \leftarrow \mathcal{O}_i^\infty \to \mathcal{O}_i^m\) are maps of \(\mathcal{O}\)-bimodules, combined with the evident fact that the operad pairing \(\mathcal{O} \circ \mathcal{O} \to \mathcal{O}\) induces a map \(\mathcal{O} \circ \mathcal{O} \to \mathcal{O}\).

\[\square\]

\textbf{Example 3.7.} For simplicity, let \(D_i(M) = \mathcal{O}(i) \wedge_{\mathcal{O}(1)} M^{\Lambda^i}\), for \(M \in R\mathcal{O}(1)\)-mod, and let \(Q = TQ\). With this notation, there is a weak equivalence \(I_{i+1}^T \simeq zD_iQ(I)\), and a commutative diagram

\[
\begin{array}{ccc}
(I_{j+1})_{i+1}^l & \xrightarrow{\mu} & I_{ij+1}^l \\
\downarrow & & \downarrow \\
zD_iQ(zD_jQ(I)) & \longrightarrow & zD_iD_jQ(I)
\end{array}
\]

where the lower left map is induced by the counit \(QzM \to M\), and the lower right map is induced by the operad structure map \(\mathcal{O}(i) \wedge \mathcal{O}(j) \to \mathcal{O}(ij)\).

\[\square\]

\textbf{3.3. Application to lifting filtrations.}

\textbf{Theorem 3.8.} Let \(I, J \in \text{Alg}_{\mathcal{O}}(R)^c\), and let \(f : I \to J^d\) be a morphism in \(\text{Alg}_{\mathcal{O}}(R)^c\). Then \(f\) induces compatible \(\mathcal{O}\)-algebra maps \(f_n : I^n \to J^{dn}\) for all \(n\), and the assignment \(f \mapsto f_n\) is both functorial and preserves weak equivalences.

\[\square\]

\textbf{Definition 3.9.} Say that a map \(f \in [I, J]_{\text{Alg}}\) has AQ-filtration\(^4\) \(s\) if \(f\) factors in \(\text{ho}(\text{Alg}_{\mathcal{O}}(R))\) as the composition of \(s\) maps

\[I = I(0) \xrightarrow{f(1)} I(1) \xrightarrow{f(2)} I(2) \to \ldots \to I(s - 1) \xrightarrow{f(s)} I(s) = J\]

\(^4\)The reader can decide if AQ stands for André-Quillen or Adams-Quillen.
such that $TQ(f(i))$ is null for each $i$.

**Theorem 3.10.** Let $f \in [I, J]_{\text{Alg}}$ have $AQ$-filtration $s$. Then there exists $\tilde{f} \in [I, J^{2^s}]_{\text{Alg}}$ such that

$$
\begin{array}{ccc}
\tilde{f} & \rightarrow & J^{2^s} \\
\downarrow & & \downarrow \\
I & \rightarrow & J
\end{array}
$$

commutes in $\text{ho}(\text{Alg}_O(R))$.

**Proof.** We work in $\text{ho}(\text{Alg}_O(R))$.

Let $f = f(s) \circ \cdots \circ f(1)$ as in Definition 3.9.

For each $i$ between 1 and $s$, there is an exact sequence of pointed sets

$$
[I(i-1), I(i)]_{\text{Alg}} \rightarrow [I(i-1), I(i)]_{\text{Alg}} \rightarrow [I(i-1), I(i)]_{\text{Alg}},
$$

and there are identifications

$$
[I(i-1), I(i)]_{\text{Alg}} \simeq [I(i-1), z(TQ(I(i)))]_{\text{Alg}} \simeq [TQ(I(i-1)), TQ(I(i))]_{\text{Mod}}.
$$

It follows that since $TQ(f(i))$ is null, $f(i)$ lifts to $\tilde{f}(i) : I(i-1) \rightarrow I(i)^2$. Theorem 3.8 then gives maps $\tilde{f}(i)_{2^s-1} : I(i-1)^{2^s-1} \rightarrow I(i)^{2^s}$. Now let $\tilde{f}$ be the composite of these $s$ maps: $\tilde{f} = \tilde{f}(s)_{2^s-1} \circ \cdots \circ \tilde{f}(1)$.

The theorem, combined with Proposition 3.5, has the following corollary.

**Corollary 3.11.** Suppose that $R$ and $O$ are connective and $J \in \text{Alg}_O(R)$ is $(c-1)$-connected. If $f : I \rightarrow J$ has $AQ$-filtration $s$, then $f_* : \pi_* (I) \rightarrow \pi_* (J)$ will be zero for $* < 2^s c$.

For more results in this spirit see [K3].

4. **Deferred proofs**

In this section we prove Propositions 2.5, 2.7, and 2.8 and Theorem 2.10. When $R = S$, so that our algebras just have the underlying structure of an $S$–module, these results can be deduced from the second author’s work, specifically [P2, Thm.1.1]. The case of a general $R$ requires a suitable generalization of that result, which we which we state as Theorem 4.4 below.

4.1. The homotopical behavior of the composition product. Fixing a commutative $S$–algebra $R$, it is useful to generalize the context slightly.

**Notation 4.1.** Let $\mathcal{P}$ be an operad in $R$-mod, i.e. a monoid object for the monoidal structure $\otimes_R$ in $\text{Sym}(R)$ defined just as in (2.1) but with $\land$ replaced by $\land_R$. We then denote by $\text{Mod}^R_\mathcal{P}$, $\text{Mod}^l_\mathcal{P}$, and $\text{Alg}_\mathcal{P}$ the associated categories of left modules, right modules, and algebras over $\mathcal{P}$ in $\text{Sym}(R)$. We endow $\text{Mod}^R_\mathcal{P}$, and $\text{Alg}_\mathcal{P}$ with the model structure as in Theorem 2.1\textsuperscript{5}.

\textsuperscript{5}Note that we do not equip $\text{Mod}^R_\mathcal{P}$ with a model structure.
Remark 4.2. If $\mathcal{O}$ is an operad in $S$-mod, then the symmetric sequence $R \wedge \mathcal{O}$, defined as $(R \wedge \mathcal{O})(n) = R \wedge \mathcal{O}(n)$ is naturally an operad in $R$-mod. It can be readily checked that there are isomorphisms of model categories
\begin{equation}
\text{Alg}_{R \wedge \mathcal{O}} \simeq \text{Alg}_\mathcal{O}(R) \quad \text{and} \quad \text{Mod}^l_{R \wedge \mathcal{O}} \simeq \text{Mod}^l_\mathcal{O}(R).
\end{equation}

To state our main technical theorem, we need the following construction.

**Definition 4.3.** Suppose given a map $f_1 : M \to N$ in $\text{Mod}_P$ and a map $f_2 : A \to B$ in $\text{Mod}_P$. Let $P(f_1, f_2)$ be the pushout of the diagram
\[
\begin{array}{ccc}
M \circ_P A & \xrightarrow{f_1 \circ_P A} & N \circ_P A \\
\downarrow_{M \circ_P f_2} & & \downarrow \\
M \circ_P B & & \end{array}
\]
in $\text{Sym}(R)$, and then define $f_1 \Boxop f_2$, the *pushout circle product* of $f_1$ and $f_2$, to be the natural map
\[
P(f_1, f_2) \to N \circ_P B.
\]

**Theorem 4.4.** Consider the composition product
\[
\circ_P : \text{Mod}_P \times \text{Mod}_P \to \text{Sym}(R).
\]
Suppose $f_2$ is a cofibration between cofibrant objects in $\text{Mod}_P$. If a map $f_1$ in $\text{Mod}_P$ is an underlying positive cofibration in $\text{Sym}(R)$, then so is $f_1 \Boxop f_2$.

Furthermore, this map will be a weak equivalence if either $f_1$ or $f_2$ is a weak equivalence.

When $R = S$, this theorem coincides with [P2, Thm.1.1], and we defer the proof in the general case to the appendix. For the purpose of proving results stated in §2, we will just need the following corollary.

**Corollary 4.5.** Let $\mathcal{O}$ be an operad in $S$-mod. Consider the composition product
\[
\circ_\mathcal{O} : \text{Mod}_\mathcal{O}(S) \times \text{Alg}_\mathcal{O}(R) \to R$-mod.
\]
Suppose $f_2$ is a cofibration between cofibrant objects in $\text{Alg}(R)$. If a map $f_1$ in $\text{Mod}_\mathcal{O}(S)$ is an underlying positive cofibration in $\text{Sym}(S)$, then $f_1 \Boxop f_2$ will be a positive cofibration in $R$-mod.

Furthermore, this map will be a weak equivalence if either $f_1$ or $f_2$ is a weak equivalence.

**Proof.** Since the functor $R \wedge \_ : \text{Sym}(S) \to \text{Sym}(R)$ sends positive cofibrations and trivial cofibrations in $\text{Sym}(S)$ respectively to positive cofibrations and trivial cofibrations in $\text{Sym}(R)$, the result follows immediately from Theorem 4.4 applied to $\mathcal{P} = R \wedge \mathcal{O}$, $R \wedge f_1$ and $f_2$. Note that the positive model
structure on $\text{Sym}(R)$ restricts on level 0 to the positive model structure on $R$-mod.

Remark 4.6. Trying to directly prove the corollary – the result we need for this paper – has led us to the more general Theorem 4.4, and, in particular, the use of operads $\mathcal{P}$ more general than $R \wedge \mathcal{O}$. This generality comes at a price: for a general $R$, the positive model structure on $\text{Sym}(R)$ is more subtle than the positive model structure on $\text{Sym}(S)$.

4.2. Proofs of results from §2.

Proof of Proposition 2.5. Let $I \to J$ be a cofibration between cofibrant objects in $\text{Alg}_{\mathcal{O}}(R)$.

Applying Corollary 4.5 to the maps

$$f_1 = * \to \mathcal{O}, \quad f_2 = * \to I$$

yields that $f_1 \square \circ f_2 = * \to I$ is a cofibration in $R$-mod, so that $I$ is cofibrant in $R$-mod.

Similarly, applying Corollary 4.5 to the maps

$$f_1 = * \to \mathcal{O}, \quad f_2 = I \to J$$

yields that $f_1 \square \circ f_2 = I \to J$ is a cofibration in $R$-mod. □

Proof of Proposition 2.7. For the first statement, we note that $B(M, \mathcal{O}, N)$ is the realization of the simplicial object $B_{\bullet}(M, \mathcal{O}, N)$, and thus will be cofibrant in $\text{Sym}(S)$ if $B_{\bullet}(M, \mathcal{O}, N)$ is Reedy cofibrant in $\text{Sym}(S)^{\Delta_{\text{op}}}$. That this is true, under our hypotheses on $M$ and $N$, is precisely the conclusion of [P2, Thm. 1.7].

Proving the second statement is similar: one sees that $B_{\bullet}(M, \mathcal{O}, I)$ is Reedy cofibrant in $R$-mod$^{\Delta_{\text{op}}}$ by noting that the proof of [P2, Thm.1.7] (and in particular that of the auxiliary [P2, Lem.5.66]) goes through if one simply replaces the very last application of [P2, Thm.1.1] by an application of Corollary 4.5. □

Proof of Proposition 2.8. First note that by Corollary 4.5 the functor

$$M \circ_{\mathcal{O}} - : \text{Alg}_{\mathcal{O}}(R) \to R\text{-mod}$$

sends trivial cofibrations between cofibrant algebras to weak equivalences, and hence, by Ken Brown’s lemma [Hir, Cor.7.7.2], also preserves all weak equivalences between cofibrant algebras.

Hence, rewriting the map

$$B(M, \mathcal{O}, I) \to M \circ_{\mathcal{O}} I$$

as

$$M \circ_{\mathcal{O}} (B(\mathcal{O}, \mathcal{O}, I) \to I)$$

one sees it suffices to show that $B(\mathcal{O}, \mathcal{O}, I)$ is cofibrant in $\text{Alg}_{\mathcal{O}}(R)$. $B(\mathcal{O}, \mathcal{O}, I)$ is the realization of the simplicial algebra $B_{\bullet}(\mathcal{O}, \mathcal{O}, I)$, viewed as a simplicial object in $R$-mod. By [HH, Prop.6.11], this agrees with the
realization of $B_\bullet(\mathcal{O}, \mathcal{O}, I)$, viewed as a simplicial object in $\text{Alg}_\mathcal{O}(R)$. Thus it suffices to show that $B_\bullet(\mathcal{O}, \mathcal{O}, I)$ is Reedy cofibrant in $\text{Alg}_\mathcal{O}(R)^{\Delta^{op}}$.

Checking this involves showing that the latching maps for $B_\bullet(\mathcal{O}, \mathcal{O}, I)$ are cofibrations in $\text{Alg}_\mathcal{O}(R)$. These depend only on $B_\bullet(\mathcal{O}, \mathcal{O}, I)$ together with its degeneracies, i.e., face maps can be ignored. From this perspective,

$$B_\bullet(\mathcal{O}, \mathcal{O}, I) \simeq \mathcal{O} \circ B_\bullet(S(1), \mathcal{O}, I),$$

where we recall that $S(1)$ is our notation for the unit symmetric sequence $(\ast, S, \ast, \ast, \ldots)$ under $\circ$. Hence, letting $\ell^\mathcal{O}_n$ and $\ell_n$ respectively denote the $n$th latching map construction on $N$–graded objects with degeneracies in $\text{Alg}_\mathcal{O}(R)$ and $R$–mod, one has

$$\ell^\mathcal{O}_n (B_\bullet(\mathcal{O}, \mathcal{O}, I)) \simeq \mathcal{O} \circ \ell_n (B_\bullet(S(1), \mathcal{O}, I)).$$

Since $\mathcal{O} \circ_\mathcal{O} : R\text{-mod} \to \text{Alg}_\mathcal{O}(R)$ is a left Quillen functor, it follows that $\ell^\mathcal{O}_n (B_\bullet(\mathcal{O}, \mathcal{O}, I))$ will be a cofibration in $\text{Alg}_\mathcal{O}(R)$ if $\ell_n (B_\bullet(S(1), \mathcal{O}, I))$ is a cofibration in $R$–mod. But the latter map is a cofibration, since it is a special case of the latching maps shown to be cofibrations in the proof of Proposition 2.7. □

**Remark 4.7.** It is straightforward to verify that the operads $R \wedge \mathcal{O}$ can also be regarded as operads in $S$–mod, so that (4.1) extends to give

$$\text{Alg}_{R \wedge \mathcal{O}} \simeq \text{Alg}_\mathcal{O}(R) \simeq \text{Alg}_{R \wedge \mathcal{O}}(S)$$

and

$$\text{Mod}^L_{R \wedge \mathcal{O}} \simeq \text{Mod}^L_\mathcal{O}(R) \simeq \text{Mod}^L_{R \wedge \mathcal{O}}(S).$$

A slightly more careful analysis shows one also has an inclusion of categories

$$\text{Mod}^L_{R \wedge \mathcal{O}} \subset \text{Mod}^L_{R \wedge \mathcal{O}}(S).$$

**Proof of Theorem 2.10 (a) and (b).** In this proof we focus on the identification $\text{Alg}_\mathcal{O}(R) \simeq \text{Alg}_{R \wedge \mathcal{O}}(S)$ so as to be able to directly apply [P2, Thm.1.1].

For part (a), note first that

$$F^R_M(I) = M \circ_\mathcal{O} B(\mathcal{O}, \mathcal{O}, I).$$

That $F^R_M(I)$ preserves weak equivalences in the $I$ variable then follows from the proof of Proposition 2.8, where it was shown both that $B(\mathcal{O}, \mathcal{O}, I)$ is a cofibrant algebra and that $M \circ_\mathcal{O} -$ preserves w.e.s between cofibrant algebras.

To see that weak equivalences are also preserved in the $M$ variable, one uses a similar argument: using the identifications (4.2) and (4.4) to change perspective to $S$–mod, one applies [P2, Thm.1.1] to any trivial cofibration

---

6It might be surprising at first that this is not an isomorphism. Unwinding definitions, one sees that given $N \in \text{Mod}^L_{R \wedge \mathcal{O}}(S)$, $N(n)$ will be a $R^n$–module. For any $N$ coming from $\text{Mod}^L_{R \wedge \mathcal{O}}$, this $R^n$–module structure must be pulled back along the canonical ring maps $R^n \to R$, which is not the case in general.
$f_1: M \to N$ in $\text{Mod}_{R \wedge O}(S)$ and the map $f_2 = * \to B(O, O, I)$. One concludes that the functor $M \mapsto F^R_M(I)$ sends trivial cofibrations to weak equivalences. One now again uses Ken Brown’s lemma.

The intuition behind part (b) comes from the observation that the formula for the composition product $X \circ Y$ of symmetric sequences, (2.1), is ‘linear’ in the variable $X$. Our official proof goes as follows. Note that homotopy fibration sequences in $\text{Alg}_O(R)$ are detected by considering them as sequences in $S$-mod. Again using the identifications (4.2) and (4.4) to change perspectives, one immediately reduces to [P2, Thm. 1.9] applied to the the operad $R \wedge O$ in $S$–modules.

**Appendix A. Proof of Theorem 4.4**

We now turn to the task of proving Theorem 4.4. If one just tries to redo all the work in [P2] with a general commutative $S$–algebra $R$ replacing occurrences of $S$, one finds that most of results generalize, with the key exception being the characterization of $S$ cofibrations in [P2, Prop. 3.10], which fails for general $R$. Here we take here a somewhat more direct approach by instead reducing the proof of Theorem 4.5 to that of [P2, Thm.1.1].

As the positive model structure on $\text{Sym}(R)$ is the projective structure induced from the positive model structure on $\text{Sym}(S)$, we have the following lemma.

**Lemma A.1.** A set of generating cofibrations for $\text{Sym}(R)$ can be obtained by applying $R \wedge -$ to a set of generating cofibrations for $\text{Sym}(S)$.

**Proof of Theorem 4.4.** The proof proceeds just as that of [P2, Thm. 1.1]. Note that the majority of the arguments in that proof are agnostic as to the category or model structure used (in particular, the filtrations of [P2, Prop. 5.24] cover $R$-mod) with the exception of the two instances where [P2, Thms 1.3, 1.4] are used. We hence indicate only the minor modifications needed.

As in [P2], in the case of $f_2$ and algebra map, one reduces to the case where $f_2: A \to B$ is the pushout of a generating cofibration. Using Lemma A.1, the latter can then be written as $\mathcal{P} \circ_R (R \wedge X) \to \mathcal{P} \circ_R (R \wedge Y)$ for $X \to Y$ a generating positive $S$ cofibration in $S$-mod. By the $\omega$-filtration argument of [P2] using [P2, Prop. 5.24] one reduces to showing that the pushout corner map in

$$
\begin{array}{ccc}
M_A(r) \wedge_{R \times \Sigma_r} (R \wedge Q^r_{r-1}) & \longrightarrow & M_A(r) \wedge_{R \times \Sigma_r} (R \wedge Y^\wedge r) \\
\downarrow & & \downarrow \\
N_A(r) \wedge_{R \times \Sigma_r} (R \wedge Q^r_{r-1}) & \longrightarrow & N_A(r) \wedge_{R \times \Sigma_r} (R \wedge Y^\wedge r)
\end{array}
$$

is a positive $R$ cofibration. Were the map $M_A \to N_A$ to be a generating positive $R$ cofibration in $\text{Sym}(R)$, one would be able to pull a $R \wedge (-)$ factor out of the pushout corner map (by Lemma A.1), and the result would then follow by [P2, Thms 1.3,1.4]. Hence, by the standard “retract of a transfinite
composition of pushouts of generating cofibrations”, it suffices to show that $M_A \rightarrow N_A$ is a positive $R$ cofibration. Following along the argument in [P2], after writing $A = \text{colim}_{\beta < \kappa} A_{\beta}$ where each map $A_{\beta} \rightarrow A_{\beta+1}$ is a pushout of a generating cofibration $P \circ_R (R \wedge X_{\beta}) \rightarrow P \circ_R (R \wedge Y_{\beta})$, one further reduces to showing, by induction on $\beta < \kappa$ that the pushout corner maps in

$$
\begin{array}{cccc}
M_{A_0} & \rightarrow & M_{A_1} & \rightarrow \cdots \rightarrow M_{A_3} \\
\downarrow & & \downarrow & \rightarrow \\
N_{A_0} & \rightarrow & N_{A_1} & \rightarrow \cdots \rightarrow N_{A_2} \rightarrow M_{A_3} \rightarrow \cdots
\end{array}
$$

are positive $R$ cofibrations and again using the $\omega$-filtration argument in [P2] this in turn reduces to showing that for each ordinal $\beta$ the pushout corner map

$$
M_{P \coprod A_{\beta}}(r) \wedge_{R \times \Sigma_r} (R \wedge Q_{r-1, \beta}) \rightarrow M_{P \coprod A_{\beta}}(r) \wedge_{R \times \Sigma_r} (R \wedge Y_{\beta}^r)
$$

$$
\downarrow
$$

$$
N_{P \coprod A_{\beta}}(r) \wedge_{R \times \Sigma_r} (R \wedge Q_{r-1, \beta}) \rightarrow N_{P \coprod A_{\beta}}(r) \wedge_{R \times \Sigma_r} (R \wedge Y_{\beta}^r)
$$

is a positive $R$ cofibration. Using the obvious analogue of Lemma A.1 for $R$ bi-symmetric sequences and [P2, Props 5.59, 5.60] (the analogues of [P2, Thms 1.3,1.4] for $\text{Sym}(S)$) just as in the argument following (A.1), one further reduces to showing that $M_{P \coprod A_{\beta}} \rightarrow N_{P \coprod A_{\beta}}$ is a positive $R$ cofibration in $R$ bi-symmetric sequences (this new notion of cofibration defined in the obvious way by analogy with $\text{Sym}(R)$). But since [P2, Prop. 5.21] identifies the $(r, s)$ level of this map with $M_{A_{\gamma}}(r + s) \rightarrow N_{A_{\gamma}}(r + s)$, the result follows by the induction hypothesis which, explicitly, says $(M_{A_{\gamma}} \rightarrow N_{A_{\gamma}})_{\gamma \leq \beta}$ is a projective cofibration.

To deal with the case of $f_2$ a general map of left modules one repeats the argument in the last paragraph of the proof of [P2, Thm. 1.1]. Finally, the case where either $f_1$ or $f_2$ are additionally weak equivalences follows by using identifications (4.2)and (4.4) to reduce the question to the $S$-mod level and then applying the monomorphism part of [P2, Thm. 1.1].

□

References


[XX] References to be still added or deleted!

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