

George Washington's Dung Repository

Introduction

Everybody knows that George Washington (1732-1799) was the father of our country, but only the most dedicated organic farmers know that he was also one of the pioneers of composting in America. In the late 18th century, when the already thin, clay topsoil was depleted from growing tobacco and cotton, the usual practice in America was to pick up and move west. But with a fabulous view of the Potomac River, plus proximity to Washington, D.C. and the shopping malls of Northern Virginia, the Washington family preferred to stay at Mount Vernon. With chemical fertilizers not yet available, Mr. Washington experimented with composting very early on. In 1787 he built a dedicated structure, which he called his “dung repository” specifically for composting. The remains of that structure have been unearthed, and in 2001 a new building was erected on the foundation of the 1787 structure. Mr. Washington's sketch of this 31 x 12 foot structure is shown in Figure 1 below.

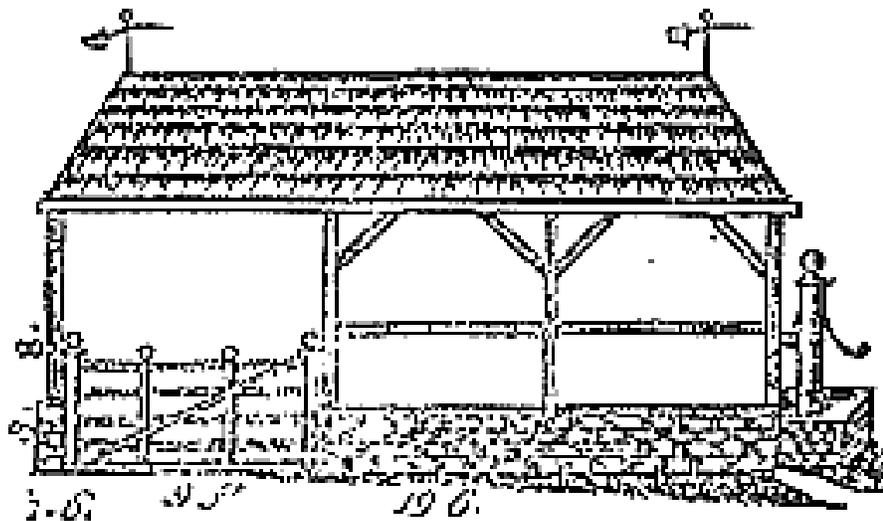


Figure 1. Side view of Washington's Dung Repository

Any practitioner knows that the composting process involves an *exothermic* chemical reaction and that it is actually possible to reach temperatures high enough inside the heap such that, instead of compost, the product is an unusable ashy material. This is where we come in. Even though Joseph Fourier (1768-1830) would have been only nine years old at the time, we will use his Law of Conduction and a few other physical principles and many assumptions to give Mr. Washington some guidance on how deep to pile the raw material.

Learning Objectives

By the time you have finished this project you should be able to:

1. Apply Fourier's Law of Conduction to one-dimensional, steady-state conduction, and, in particular, understand the sign convention.
2. Derive the governing, ordinary differential equation for steady-state conduction with volumetric heating in a one-dimensional geometry,

3. Formulate appropriate thermal boundary conditions and explain them physically,
4. Determine the appropriate governing parameters and estimate their values,
5. Solve the governing differential equation analytically and apply appropriate boundary conditions,
6. Set up a spreadsheet to evaluate your analytical solution at a number of points and display your solutions,
7. Explain all assumptions used in the model,
8. Check the validity of your solution,
9. Test parameters readily and on the basis of your calculations provide guidance to the client, in this case, the soon-to-be President of the United States.

Formulation of the Problem

We assume that a one-dimensional model is valid, that is, the length and width of the proposed depository are long compared to the depth we ultimately recommend. We also assume that conduction is the predominant heat transfer mode. (The layer may in fact be porous and thus have air and/or water percolating through it, carrying heat in the process.) We will also assume steady-state conditions, which is really a stretch, because a layer as thick as one might envision here takes a fairly substantial amount of time to heat to a steady condition. In addition dedicated composters know that for best results the compost should be turned periodically.

Before proceeding much farther we must select a coordinate system. Three possibilities are shown in Figure 2 below.

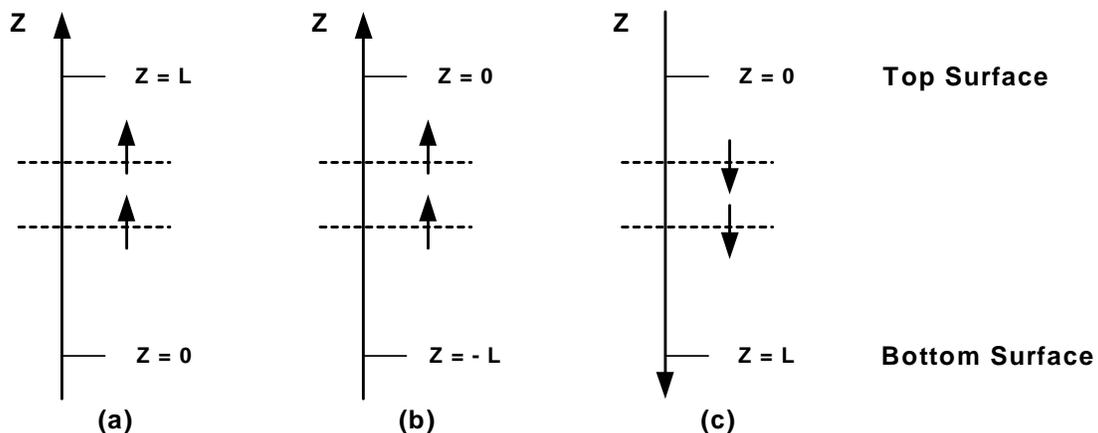


Figure 2. Three Possible Coordinate Systems.

The first two systems are really identical except in where the boundary conditions are applied. An earth scientist, who stands at the surface and is most interested in things between him and the center of the earth, might prefer the third system. We'll select the first system, but the student should go through the exercise of deriving the governing equation and the boundary

condition for the third system to ensure that he or she really understands the sign conventions used. We first write a steady-state heat balance for the hypothetical thin layer indicated by the dotted lines:

$$q_{cond\ in} - q_{cond\ out} + \dot{q}_{generated} = 0, \quad (1)$$

that is, in steady-state heat conducted in through the lower boundary plus heat generated within the layer must be balanced by heat conducted out through the upper boundary. Note that a conduction flux in the positive z direction through the lower face is a net gain to the layer, while conduction flux in the positive z direction through the upper face represents a loss to that layer. Then using Fourier's Law of Conduction to represent the conductive fluxes and assuming a layer of thickness Δz , we obtain:

$$-k A \frac{dT}{dz} - \left(-k A \frac{dT}{dz} + \frac{d}{dz} \left(-k A \frac{dT}{dz} \right) \Delta z \right) + \dot{q}''' A \Delta z = 0. \quad (2)$$

After shrinking Δz to 0.0, canceling the area (A) and assuming the thermal conductivity (k) is uniform, Equation 2 simplifies to:

$$\frac{d^2 T}{dz^2} + \frac{\dot{q}'''}{k} = 0 \quad (3)$$

This second-order ordinary differential equation needs two boundary conditions. Let us apply an adiabatic condition at the bottom $\left(\frac{dT}{dz} = 0 \right)$ and a convection condition $\left(-k \frac{dT}{dz} - h(T - T_\infty) = 0 \right)$ at the top of the layer ($z = L$). You must justify these choices later.

(Note: Since no heat can leave through the adiabatic bottom, the two expressions for the heat transfer from the top in the previous equation (coming from Fourier's Law applied on the solid side of the compost-air interface and Newton's Law of Cooling applied on the air side) are both equal to the total heat generated (per unit area) in the layer.)

Implementation

1. Solve the differential equation above (on paper), applying the boundary conditions suggested. Note that for purposes of this solution L, the depth of the layer, is fixed.
2. Set up a spreadsheet to evaluate and view your solution for the both the temperature and heat flux as a function of depth ($0 \leq z \leq L$). The solutions for both, especially the later, are quite simple and could readily be done using cell formulae, but instead create Visual Basic for Applications (VBA) functions for both of them. Your spreadsheet should allow for easy input of the volumetric heat generation rate (\dot{q}'''), the convective heat transfer coefficient (h) at the exposed surface, the depth of the layer, the thermal conductivity of the composting material (k) and the temperature of the air in contact with the surface of the layer (T_∞). Divide the depth into 20 segments, i.e., you will evaluate and plot your analytical solution at 21 points through the thickness of the layer.

NOTE: You are finding numerical values corresponding to an analytical solution at a finite number of points here. This operation is **not** to be confused with a *numerical* solution. The latter generally involves the *discretization* of the governing equation, approximation of derivatives and subsequent solution of algebraic equations at a finite number of points. You are applying what you learned in ordinary differential equations, where generally analytical solutions are stressed.

3. As a start, some reasonable values of the inputs are: $\dot{q}''' = 8 \frac{W}{m^3}$, $h = 25 \frac{W}{m^2 K}$, $k = 0.1 \frac{W}{mK}$, $T_\infty = 21^\circ\text{C}$ (294K), $L = 1.0\text{m}$. A temperature of 65°C (338K) is considered to be an approximate upper limit for the temperature allowable in the layer (see Reference 1).

4. As a check, assume now that all the heat generation takes places at the bottom of the layer. In other words you have a source of heat there producing an amount equal to that generated through the whole thickness of the layer. What would the peak temperature be if all the heat had to be conducted through the entire thickness to reach the exposed surface? What would the temperature and heat flux plots look like now?

Things To Think About:

1. What is the best way to display your solution, both the temperature distribution and the heat flux, for your client?
2. Why does the heat flux take the shape that it does?
3. How can you check that the numbers you compute are correct? You should find several methods.
4. How does the heat flux at the top surface (W/m^2) relate to the volumetric heat generation rate (W/m^3)?
5. Why does the magnitude of the slope of the temperature versus depth curve become greater as you approach the surface?
6. How does the surface heat flux (W/m^2) you compute compare with, for example, the solar flux on a sunny day?
7. How does the volumetric heat generation rate (W/m^3) compare with, for example, that in the core of a commercial nuclear reactor, or that when you microwave a casserole?
8. How important is the value of the surface convective heat transfer coefficient that you use in determining the maximum temperature in the layer?
9. How does our choice of thermal boundary conditions force the maximum computed temperature to occur at the bottom of the layer? Is another boundary condition more appropriate?

References

1. Datta, A.K., *Biological and Bioenvironmental Heat and Mass Transfer*, Dekker, New York, 2002, p 67.

2. Pogue, D.J. and Arner, R., George Washington, The Revolutionary Farmer: America's First Composter, <http://www.cityfarmer.org/washington.html> , Published by City Farmer, Canada's Office of Urban Agriculture.
3. Fourier, J., *The Analytical Theory of Heat*, translated by Alexander Freeman, 1878, Dover Edition, 1955.