Transient Conduction at the Interface between Two Materials

Introduction

In this exercise we will use the finite-volume method (FVM) to solve a transient, 1-D conduction problem that models the contacting of two dissimilar materials having different initial temperatures. Let us assume that one of the materials represents your finger, while the other is one of the several inorganic materials we want to test. We will start with the temperature of the material being tested high enough that you would expect to experience discomfort and possibly a burn if you touched it for long enough. Then we will run the calculation for a brief period of time and compare results for three materials.

Implementation

A sketch of the 1-D geometry is shown below. For a start let us take the grid spacing as $\Delta x = .0001$ m and place 20 nodes in the section representing your finger and 20 in the other material. (You should design your spreadsheet or program so that these numbers can be changed easily.)

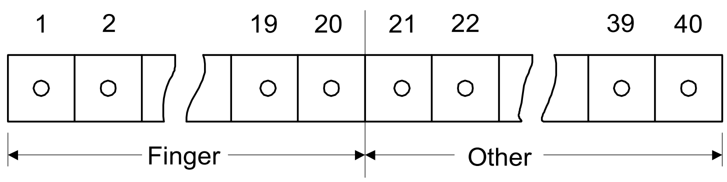

Figure 1 Nodal Configuration

The appropriate density, specific heat and initial temperature may be readily assigned to each node. The assignment of the thermal conductivity is slightly complicated by the fact that, as explained later, we need the values at the cell boundaries rather than at their centers. Let us adopt the convention that "k_i" represents the conductivity at the right edge of cell "i"; then "k_{i-1}" will be the value to its left. The assignment of values is then straightforward, except at the interface between the two materials (k_{20} for the grid arrangement shown above). Here it is probably appropriate to say that you have a layer of flesh of thickness $\Delta x/2$ in series with a layer of thickness $\Delta x/2$ of the material being tested and then assign the appropriate equivalent value. We are assuming here that there is no "contact resistance" between the two materials; you could readily include a non-zero value of contact resistance as a third resistance in series at this location. (Indeed, if you were doing a real physical experiment rather than this computer simulation, you might deliberately add some extra thermal resistance between yourself and the hot surface by wetting your finger.)

With properties denoted as 1-D vectors (in fact, the density and specific heat may be combined into one variable since they always appear as a product), the transient heat balance equation may be written in an "explicit" form as:

$$(\rho c_p)_i \frac{T_i^{p+1} - T_i^p}{\Delta t} \Delta x\ A = -\ k_{i-1} A \frac{T_i^p - T_{i-1}^p}{\Delta x} - (-) k_i A \frac{T_{i+1}^p - T_i^p}{\Delta x}$$

That is, the time rate of change of stored thermal energy is equal to the difference between the energy conducted in through the left side and that conducted out through the right. Here the superscript p refers to the current time level, while p+1 refers to the advanced (unknown) time level. One negative sign in each of the conduction terms on the right comes from Fourier's Law; the other one in the second term comes from our convention that a positive flux into the left side is a net addition to that cell, while a positive flux through the right side represents a loss to that cell. This equation, when solved for the unknown T_i^{p+1}, gives a "prescription" which can be used to predict temperatures in terms of known values. Note that the cross-sectional area (A) cancels out.

Upon studying the above equation and our nodalization (Figure 1), it should be evident why the heat capacity multiplying the storage term is evaluated at cell centers while the thermal conductivity that appears in the two flux terms is evaluated at the edges. By evaluating the thermal conductivity at the edges, we can be sure that a term having equal magnitude and opposite sign appears in the heat balance we apply at the next-door neighbor - thereby ensuring that our scheme conserves energy exactly. Some authors will emphasize this placement of the thermal conductivity at the cell edges by designating it as $k_{i-1/2}$, $k_{i+1/2}$, etc., but unfortunately computers do not like fractional subscripts.

Start with an initial flesh temperature of 30°C and an initial material temperature of 300°C. Take the outer boundaries to be fixed for all time at these values. Especially for a high thermal conductivity material like cast iron, this assumption may be faulty. If you choose to do an explicit calculation as derived above, you will need to compute the allowable time step (Δt). This value can be determined by solving the above equation for the unknown T_i^{p+1} and then requiring that the coefficient of the central term $\left(T_i^p\right)$ be greater than or equal to 0. (See any heat transfer text.) The result you get from this analysis reduces to the usual restriction on timestep r (= Fo) $\leq \frac{1}{2}$ for a slab of a single material. Here the limiting timestep could arise in the finger nodes (1-19), in the "other" material (nodes 21-40) or at either of the hybrid nodes adjacent to the interface (20 and 21). You will need to check all four of these possibilities and then apply the minimum of the four. The particular cell that limits the timestep is not necessarily the same one for all three materials. For the soapstone and shuttle tile runs the limiting timestep will be pretty reasonable; for the cast iron it will be very small, and you might be well advised to switch to an implicit formulation.

Material properties are supplied in the following table:

Material	Density $\left(\frac{kg}{m^3}\right)$	Specific Heat $\left(\frac{J}{kg \cdot K}\right)$	Thermal conductivity $\left(\frac{W}{m \cdot K}\right)$
Flesh	1000	4181	0.37
Cast Iron	7608	400	80.2
Soapstone	2793	971	2.15
Space Shuttle tile	144.2	878.6	.06

In lieu of better information, values for water are listed for the density and specific heat of flesh; the thermal conductivity is a value for skin. (Since we are performing this experiment on the computer rather than for real, it might be more appropriate to use the values tabulated in the appendix of most textbooks for chicken meat!) For most of their approximately 8 cm. thickness, space shuttle tiles are made out of a material with the properties given in the table. (It is almost like Styrofoam[tm].) There is, in fact, a thin, almost egg-shell-like layer on the outside which has a much higher thermal conductivity than that given in the table. We will neglect that layer in this analysis.

Run your calculation out to an elapsed time of 0.1 seconds, which is probably a reasonable

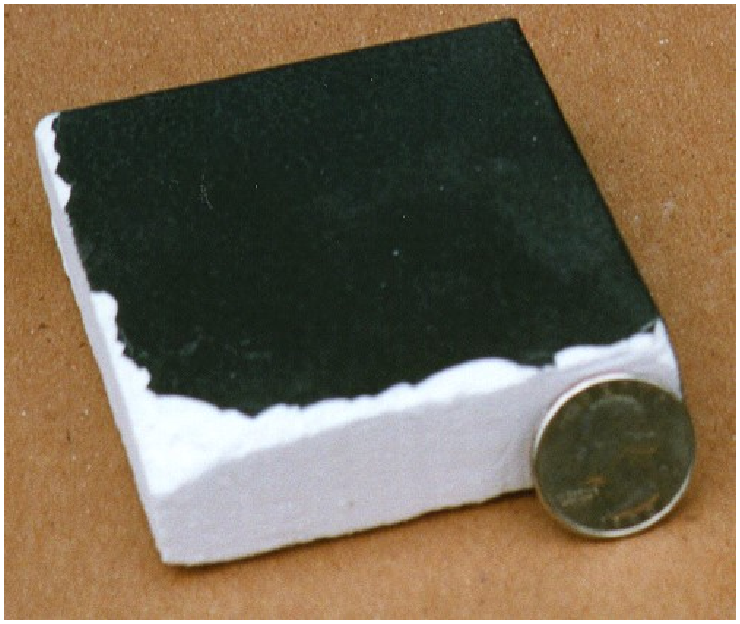

Figure 2. Sample of space shuttle tile. This piece is from the windward (under) side and coated with a high emissivity coating to enhance radiative losses.

estimate of how long it would take you to respond if your hand were being burned. Plot your results (temperature vs. position at time = 0.1 secs.) for the three materials and comment. Is the assumption that the temperatures at the left end of the "finger" and the right end of the other material do not change during this transient good for all three cases? Can you explain why it is so much more comfortable to step with your bare feet onto a rug instead of on bare tile or porcelain at the same temperature?

Verification

As long as the elapsed time is insufficient for the transient to propagate to the outer edges of the computing region we have chosen, then an analytical solution based on two semi-infinite solids placed in contact is valid. From that solution we can solve for the interface temperature in terms of the initial temperatures of the two bodies and their thermal properties[1] [1]:

$$T_s = \frac{(k\rho c)_A^{1/2}\, T_{A,i} + (k\rho c)_B^{1/2}\, T_{B,i}}{(k\rho c)_A^{1/2} + (k\rho c)_B^{1/2}}$$

[1] The square root of the product of the thermal conductivity (k) and the volumetric heat capacity (ρc_p) of a material (as appears in the solution for the interface temperature) is sometimes called the thermal *effusivity* and may occasionally be found tabulated along with other thermal properties.

From that solution you determine the contact temperature for each of the three cases (surprisingly enough it is **not** a function of time) and compare to what your program predicts. For space shuttle tile and the soapstone, the 0.1 second duration of the transient as suggested earlier is sufficiently short that good agreement between this analytical prediction and the numerical solution will be found. Unless you make your cast iron layer thicker than suggested, you should find that the semi-infinite approximation is not valid that long into the transient.

Reference

Incropera, F.P. and DeWitt, D.P., *Fundamentals of Heat and Mass Transfer*, 5th Ed., Wiley, NY, 2002.

touch 9/17/12 updated 4/18/2018