

Methodology brief

Introducing the J.P. Morgan Implied Default Probability Model: A Powerful Tool for Bond Valuation

- This brief presents a dynamic default probability framework and its applications to emerging markets bonds
- Default probabilities are the purest indicator of an issuer's implied credit risk
- Unlike stripped spreads, default probabilities are not affected by a bond's specific characteristics like liquidity, cash-flow complexities, embedded options
- Default probabilities find many useful applications:
 - Revealing and identifying a bond's relative value beyond the capabilities offered by stripped spread
 - Valuing the convexity of a bond
 - Valuing embedded call and put options
 - Estimating the implied price assigned by the market to such factors as liquidity, repo specialness, and complexity of cash flows
 - Estimating fair prices for bonds in different currencies from the same issuer
 - Estimating fair new issue prices
- Two assumptions are key to the determination of default probabilities: recovery value and volatility of default probabilities
- The appendix contains technical details key to the calculation of implied default probabilities

I. Introduction

In this brief, we present another tool to assess the value of emerging markets bonds and to identify – and quantify – the cheapness or expensiveness of a bond.

Traditionally, stripped spreads have been used to assess the implied riskiness of an issuer. However, stripped spreads are affected by several factors, including, but not limited to, credit risk. The true underlying credit risk is captured by the implied default probabilities. In fact, bonds trading at similar stripped spreads may imply different default probabilities. Therefore, stripped spreads may present a distorted measure of the riskiness of an issuer. Implied default probabilities provide us with a

more accurate estimate.

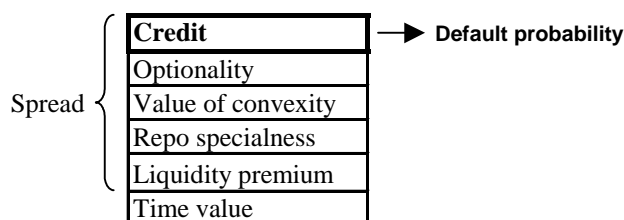
In order to extract the true probabilities of default from the market price of a bond, we have to strip all of the value unrelated to the credit risk from its market price. In principle, the price of a bond captures its time value, value of embedded options, value of convexity, price adjustment due to repo specialness, liquidity premium, and the core value of credit.

In contrast to the stripped spread, behind which lie all the components listed above except time value, default probability reveals the true credit risk of the bond (see Exhibit 1).

Exhibit 1

Default probability is only related to the credit value

Components of the price of a bond



Two key assumptions in our model are recovery value and volatility of default probabilities. After making these two assumptions we are able to generate a dynamic default process that has many applications. For example, we can calculate the value of convexity and embedded call and put options. Moreover, once a term structure of default probability has been derived from liquid, non-special, plain vanilla bonds, we can estimate the value of liquidity, repo specialness, and complexity of cash flows implied by the bond's market price.

Finally, using the term structure of default probabilities

We would like to thank Neil Yang from Derivatives Research for his contribution to this methodology brief.

All valuations and exhibits in this publication are based on prices as of September 19, 2000.

we can estimate the fair price of new issue bonds and the fair price of bonds of the same issuer denominated in any currency.

Technical details key to the calculation of default probabilities appear in the appendix.

II. First step: determining static default probabilities

Static probability of default can be explained with a simple one-period zero coupon bond. For such a bond, there are only two possible outcomes upon maturity:

- (1) default with probability P , in which case the bond pays recovery value R ;
- (2) no default with probability $1-P$, in which case the bond pays \$100 (see Exhibit 2).

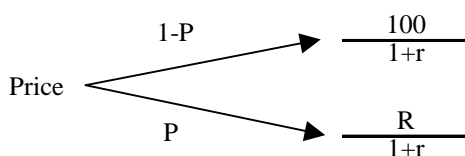
The price of the bond is simply the expected value of the two possible outcomes discounted at the risk-free interest rate r .

Conversely, given the market price of the bond and a recovery value assumption, we can derive the implied default probability P assuming the risk-free interest rate is known.

Exhibit 2

Price and recovery value imply default probability

One-period zero coupon bond



In the general case of multiperiod bonds, we can simply extend the example above, several periods out until maturity, while assuming the probability of default remains constant, or “static,” throughout the life of the bond. Given the market price of the bond and a recovery value assumption, we can extract the implied, static default probability that prevails from now until the bond’s maturity.

Finally, given that at this stage everything in our model is static, we also assume risk-free interest rates in the future are static and equal to the implied forward risk-free rates. We use swap rates used to determine the term structure of risk-free interest rates.

We assume that recovery value is a fixed percentage of the bond’s principal. There are generally three alternative assumptions on recovery value: (1) recovery value is a certain percentage of the promised cash flows; (2) recovery value is a certain percentage of the market value of the bond immediately prior to default; (3) recovery is a certain percentage of the bond’s principal.

In this piece, we choose recovery value assumption (3). This is primarily due to the fact that if default happens, all claims by bondholders on the sovereign issuer in default collapse to the principal and accrued interest, regardless of specific features of the bonds. In many of the past emerging market restructurings – including several Brady plans and the two Vnesh restructurings – the claims were restructured by offering a restructuring of the principal that was homogeneous across the different claims. This type of restructuring would be in line with our recovery value assumption.

It is important to note, however, that our recovery value assumption has found only a partial application in the case of the recent Ecuadorean debt exchange. In the specific case of the two Ecuadorean Eurobonds – EC Rep 02s and 04s – and EC IE bonds, principal claims were exchanged for the same amount of new bonds. Still, our recovery value assumption would have not explained the treatment of the other bonds.

We opt for a recovery value of 20% of the principal.

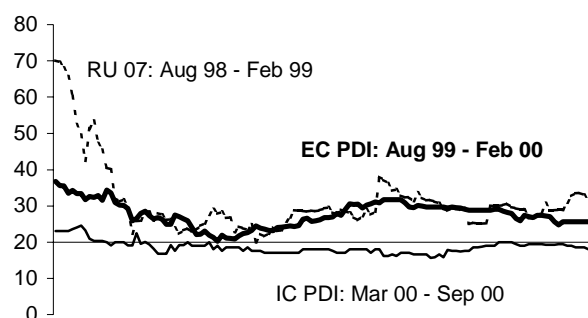
For amortizing or capitalizing bonds, the recovery value is assumed to be 20% of the remaining principal. To determine this value we have analyzed the performance of several bonds at times when a sovereign issuer was in or near default. Specifically, we have looked at Ecuador PDIs, Russia 07s, and Ivory Coast PDIs.

Exhibit 3 shows that these bonds had traded around \$20 when their sovereign issuers were in or near default.

Exhibit 3

Bonds in or near default traded around \$20

Settlement price (dollar) vs time



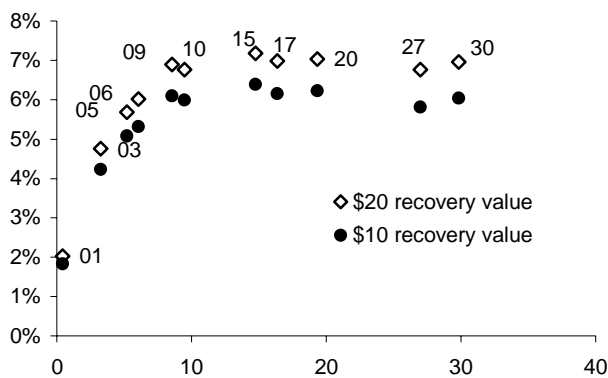
Source: J.P. Morgan

Exhibit 4 shows the annualized static probabilities of default for Argentine Eurobonds under recovery value assumption of \$10 and \$20, respectively. As can be seen, changing the recovery value from \$10 to \$20 affects the level, but not the shape of the implied default probability curve. The relative relationship among the default probabilities implied by different bonds are not affected if we allow recovery value to change within a reasonable range.

Exhibit 4

The recovery value assumption affects the level, but not the shape of default probability curves

Argentina: static prob of default (annual) vs. bond maturity (yrs)



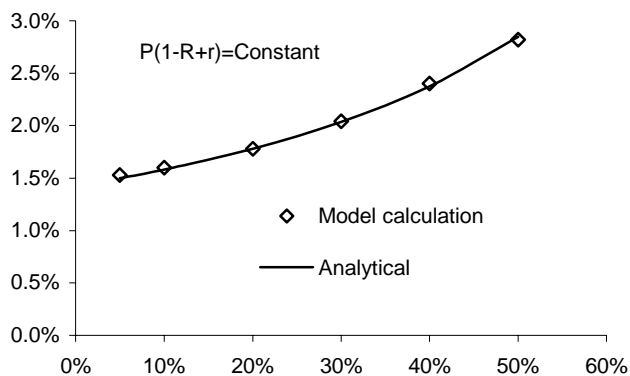
Source: J.P. Morgan

In fact, the fixing of recovery value can be easily relaxed as the default probability under one recovery value is related to the default probability under another recovery value through a simple analytical relationship. Specifically, the product of (1) default probability and (2) one minus recovery value plus interest rate is approximately a constant. Exhibit 5 shows that this

Exhibit 5

Default probability is related to recovery value through a simple analytical relationship

Default prob (annual) vs recovery value (Mexico UMS 07)



Source: J.P. Morgan

relationship is a very good approximation. We calculated the implied default probabilities of UMS 07s under various recovery values, and plotted these probabilities and the ones derived from the simple analytical relationship on the same chart. An explanation for this relationship is given in the appendix.

III. Second step: determining dynamic default probabilities

The static probability of the default model assumes that the probability of default remains unchanged throughout the life of the bond. However, the riskiness of a Sovereign issuer changes over time. As such, a realistic default process should allow the default probabilities to fluctuate. Here we extend the concept of *static* probability of default to *dynamic* probability of default by allowing probability of default to follow a diffusion process with a term structure of volatility.

Similar to the static case, the market price of a bond and a recovery value assumption imply a unique, average dynamic probability of default. In other words, there is a unique average level around which probability of default fluctuates that matches the expected present value of the bond with its market price.

Finally, we replace the static forward interest rate assumption with a dynamic, two-factor Heath-Jarrow-Morton interest rate process.

Ideally, we should be able to obtain forward-looking implied volatility of default probability from the bond options markets. However, liquid bond options exist only for a group of selected bonds in a few emerging market countries. In addition, options with maturity beyond one year are very rare. As such, the options markets does not provide us the information we need to calculate forward price volatility.

As an alternative, we derive the term structure of default probability volatility from historical price volatilities. As the appendix shows, given the price volatility of a bond, there exists an equivalent default probability volatility. From a set of bonds with different maturities from the same sovereign issuer, we can obtain a set of corresponding default probability volatilities for each maturity.

Each of these volatilities is the average, constant volatility that prevails until maturity of the respective bond. We can then extract the term structure of default probability volatility that is implied by these average volatilities of bonds with different maturities.

Dynamic probability of default is higher than the static one due to the convexity effect. Similar to the case of positive convexity of price to yield, bonds exhibit positive convexity of price-to-default probability (the appendix provides a proof for the positivity of this convexity).

As default probability increases, the price of a bond falls, and vice versa. However, a symmetric rise and fall in default probability leads to an asymmetric fall and rise in price – the price rises more than it falls. As such, in order for the average price (or expected price) to be equal to the market price, the probability of default has to be higher – on average – with volatility than without it.

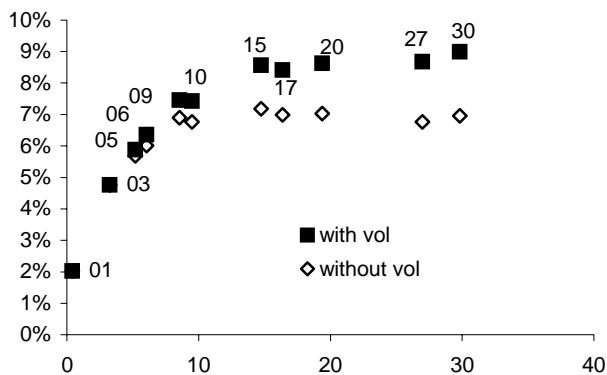
Exhibit 6 shows how the introduction of nonconstant default probability and risk-free interest rates contribute to the rise of implied default probabilities. In general, the increase in implied default probability is almost entirely due to the volatility of default probability. The volatility of risk-free interest rates has only a secondary impact. As Exhibit 6 also shows, the longer duration bonds tend to look expensive without volatility, whereas with volatility, the bonds no longer look expensive.

The default probability curve takes an upward sloping shape, more in line with the market perception of risk.

Exhibit 6

Volatility increases implied probability of default

Argentina: Probability of default (annual) vs. bond maturity (yrs)



Source: J.P. Morgan

An important by-product of this observation is that we can calculate the value of convexity. The rise of dynamic probability of default over the static one is due to the introduction of volatilities – default probability volatility and interest rate volatility.

If we remove these volatilities, the dynamic probability of default will lead to a price lower than the market price.

Convexity explains the difference between the two prices. Exhibit 7 shows the value of convexity for Argentine Eurobonds calculated in this way. Long-maturity bonds typically have a higher value of convexity than short maturity bonds.

We believe these values should be regarded as indications of the potential value of a bond’s convexity since investors may not be able to fully monetize them due to market frictions and imperfect timing of transactions.

Exhibit 7

Volatility helps us value convexity

Value of convexity of Argentine Eurobonds (bps)

Rep 05	14
Rep 06	30
Rep 09	54
Rep 10	58
Rep 15	107
Rep 17	111
Rep 20	121
Rep 27	125
Rep 30	133

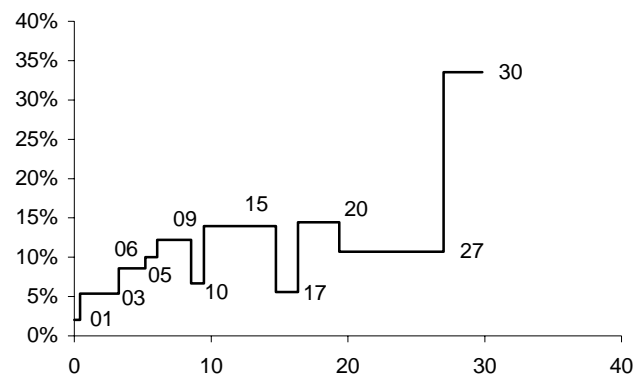
Source: J.P. Morgan

At this stage, forward default probabilities can be calculated. The probabilities we talked about so far are the independent probabilities idiosyncratic to each bond. One bond may imply a different probability than another for the overlapping life time of the bonds. A term structure of probability of default, or forward probability of default, can be bootstrapped from these independent probabilities to generate a consistent picture of default probability. Exhibit 8 shows the bootstrapped probabilities of default from Argentine Eurobonds. This term structure has a step-wise shape because the forward

Exhibit 8

Term structure of default probability is generally upward sloping with volatility

Forward prob of default (annual) vs starting year (Argentina)



Source: J.P. Morgan

probabilities of default between the maturity of any two nearby bonds are assumed to be flat. The term structure of default probability with volatility is generally upward sloping.

This term structure of default probability and the term structure of default probability volatility allows us to generate a default process to value embedded options a bond may have. More discussions about our approach to the valuation of these options can be found in the appendix. In short, we account for the value of options when we extract implied default probabilities from bonds with embedded options.

Finally, we would like to mention that with volatility numerical simulations indicate that the approximate relationship between default probability and recovery value still holds. In fact, this relationship is a fair approximation even for bootstrapped probabilities under various recovery values within reasonable range.

IV. Applications of our implied default probability model

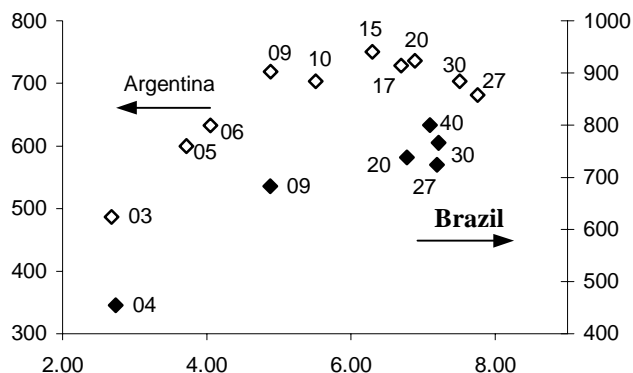
Given the fundamental nature of the probability of default, we can find several important applications for it.

(i) *Relative value analysis*: An analysis of implied default probabilities can shed light on apparent anomalies of spread curves. For example, at first glance, the slope of the Argentine and Brazilian spread curves – particularly the very long end – seems questionable, as bonds with very small duration differences trade at very different spread levels (see Exhibit 9).

Exhibit 9

The long end of spread curves looks puzzling ...

Stripped spread (bps) vs spread duration (yrs)



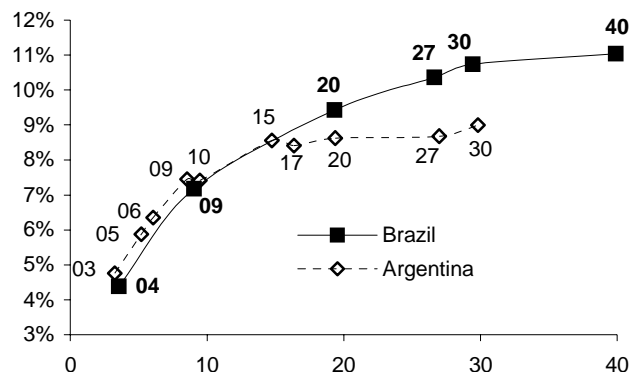
Source: J.P. Morgan

However, when we analyze dynamic probabilities of default for Argentine and Brazilian Eurobonds, both curves appear quite smooth, with the only exception being the Argentine Republic 15s and Republic 30s that seem genuinely cheap (see Exhibit 10).

Exhibit 10

...but Brazilian and Argentine Eurobonds have a fairly smooth dynamic default probability curve

Probability of default (annual) vs. bond maturity (yrs)



Source: J.P. Morgan

Clearly, many of the kinks of the spread curve are explained by the default probability curve, which has a much smoother profile. However, we believe irregularities in the default probability space warrants more investigation for relative value opportunities.

(ii) *Value and compare bonds denominated in different currencies from the same issuer*: Default probabilities derived from dollar-denominated Eurobonds can be used to assess the cheapness (or richness) of other *pari-passu* bonds – namely nondollar-denominated Eurobonds and restructured Brady bonds. Normally, all Eurobonds and Brady bonds from a sovereign issuer are ranked at least *pari-passu* with one another, and linked by cross-acceleration clauses. As such, default risks should be the same for bonds issued in different currencies.

We have seen a surge of euro-denominated bonds since the introduction of the euro. While these bonds are typically less liquid than their dollar-denominated counterparts, we can use the implied default probabilities of dollar-denominated bonds to obtain an indicative reference price for the illiquid bonds.

Among the liquid euro-denominated bonds, the prices implied by these probabilities can be compared with the market prices for relative value analyses.

For example, Exhibit 11 shows a selection of euro-denominated bonds. The Mexican bond seems cheap, while the Argentine and Turkish ones are expensive relative to dollar-denominated bonds.

Exhibit 11

Overall, euro denominated bonds seem expensive

	Market Price	Implied Price
AR 9% Jun 03 EUR	100.75	98.80
MX 7 1/2% Mar 10 EUR	96.75	97.37
TR 9 1/2% Mar 04 EUR	103.75	101.00

Source: J.P. Morgan

Although our model suggests the euro-denominated bonds are expensive relative to dollar-denominated bonds, capturing the relative expensiveness of euro-denominated bonds could be challenging. A first obstacle is the illiquidity of euro-denominated bonds, both in the cash and repo markets. In addition, the differential in pricing between euro- and dollar-denominated bonds represents a relative value opportunity only when the correlation between sovereign spreads and FX rates is taken into account.

Indeed, there could be a level of correlation between the two that would explain the current pricing. Conversely, there are levels of correlation between sovereign spreads and FX rates that would make the relative value opportunity even more attractive. Our analysis does not account for this correlation, which in a way is similar to assuming that the spread movements and FX rate movements are not correlated.

(iii) Quantifying the premium associated with complex cash flows: Another recurring issue in emerging markets is whether restructured Brady bonds are cheap relative to Eurobonds. Brady bonds typically have more complex cash-flow structures than Eurobonds, for which markets seem to charge a premium. Brady bonds normally rank *pari-passu* with all other external indebtedness of a sovereign issuer, including Eurobonds. As a result, we can apply Eurobonds' default probabilities to Brady bonds to calculate the premium charged by the market for the complexity of the Bradys' cash flows.

In the following example, we apply the default probability and its term structure of volatility implied in Argentine and Brazilian Eurobonds to their respective Brady bonds; at the same time, we take into account the callability of the Brady bonds.

We find that current market prices imply a hefty premium for the complex, cash-flow structures of some Bradys.

A revealing way to highlight the cheapness of a Brady bond is to calculate its fair price using default probabilities implied by the Eurobond curve but assuming recovery values equal or lower than those used for Eurobonds (\$20). As Exhibit 12 shows, even if the recovery value for DCBs were \$10, the market prices of Brazil DCBs would be still lower than the calculated fair prices. On the other hand, EIs seem to be expensive relative to the Eurobonds. NMBs seem fair to the Eurobonds with little premium for the complexity of their cash flows. The market price of Argentina FRBs also implies a high premium for the complexity of cash flow. Although a simple spread curve may give us a clue which Brady bond is cheaper relative to Eurobonds (see Exhibit 13), our default probability model is capable of quantifying the level of cheapness of each Brady while taking into account all the features of the selected bonds such as convexity, amortizations, and embedded options.

Exhibit 12

BR DCBs and AR FRBs appear attractive to Eurobonds

Price of Bradys under various recovery values, subjecting to the same default prob and default prob volatility as Eurobonds

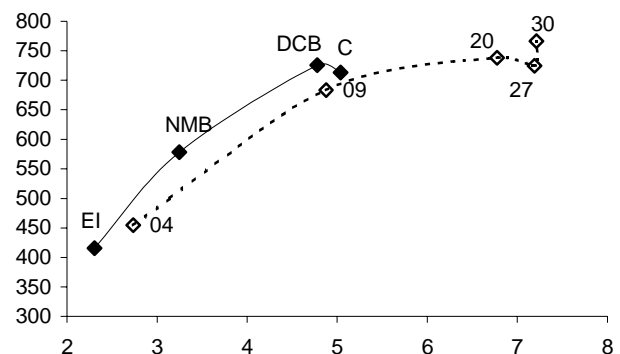
	Market price	Price under recovery value of	
		\$20	\$10
BR EI	94.13	92.78	91.68
BR NMB	87.13	87.53	85.70
BR DCB	75.75	78.79	75.76
BR C	75.50	77.87	74.60
AR FRB	91.75	93.97	93.01

Source: J.P. Morgan

Exhibit 13

The spread does not always tell the full story about relative value between Bradys and Eurobonds

Stripped spread (bps) vs spread duration (yrs)



Source: J.P. Morgan

(iv) *Quantifying the premium associated with repo specialness*: Pricing a bond on repo special using default probabilities implied by bonds that are not on repo special shows us the value the market assigns to the repo specialness. For example, Brazil Republic 08s are on repo special, and trade at \$89.0. If we subject Republic 08s to the riskiness implied by Republic 01s, 04s, and 09s, then the price of 08s would be \$85.1. Therefore, it appears that the markets assign an extra \$3.9 to Republic 08s due to their specialness in the repo markets. Our model allows investors to assess whether this is a fair premium to be paid in exchange for the favorable funding rate.

(v) *Estimating the fair value of new issues*: New-issue bonds can also be priced using implied default probabilities of existing bonds. If the maturity of new-issue bonds is longer than any existing bond, we can extend the flat forward probability of default at the unchanged level and the default probability volatility along its natural shape (see the appendix for the analytical approximation for the default probability volatility).

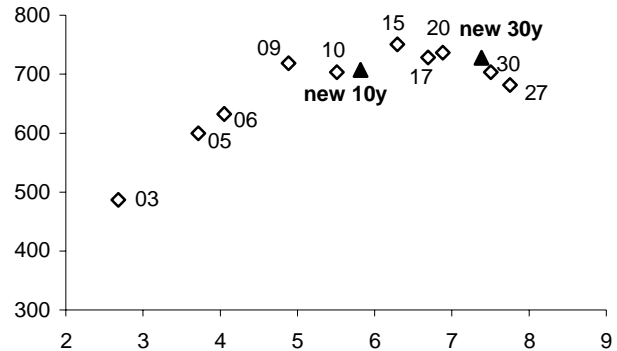
Specifically, for any new-issue bond, be it a fixed rate bond or a floater, we can always find the coupon or spread so that the price of the bond equals the issue price. We can, in turn, calculate the spread of the bond and obtain the new-issue spread. Exhibit 14 shows the spread levels of two hypothetical new issue 10-year and 30-year Argentine Eurobonds priced at par.

In this example, the new 30-year bond seems cheap versus Republic 27s and the existing Republic 30s when looking at the spread curve. But it is its position on the default probability curve that matters. As Exhibit 15 shows, the new-issue fair price calculated by our default probability model places these bonds right on the implied default probability curve.

Exhibit 14

Default probabilities of existing bonds help determine the spreads of new issue bonds priced at par

Argentina: stripped spread (bps) vs. spread duration (yrs)

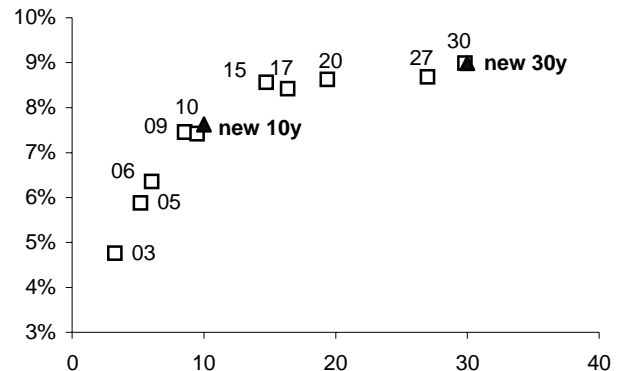


Source: J.P. Morgan

Exhibit 15

New bonds appear fairly priced on the default probability curve

Probability of default (annual) vs. bond maturity (yrs)



Source: J.P. Morgan

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Appendix

How We Calculate the Implied Default Probabilities

We start from the case of constant probability of default.

This can be modeled with a Poisson process with a parameter λ . Under this process, the probability of no default from time zero until time t is $\exp(-\lambda \cdot t)$. The annualized probability of default is $1 - \exp(-\lambda)$, which is approximately λ when λ is not too large. As such, we will loosely refer λ as the annualized probability of default, or, simply, probability of default.

Under the assumption of recovery value R for one dollar of principal, the market price of a bond paying annual coupon of C should be

$$\begin{aligned} P &= \sum_{i=1}^n C \frac{1}{(1+r)^i} e^{-i\lambda} + \frac{1}{(1+r)^n} e^{-n\lambda} \\ &\quad + R \sum_{i=1}^n \frac{1}{(1+r)^i} [e^{-(i-1)\lambda} - e^{-i\lambda}] \\ &= \sum_{i=1}^n C \frac{1}{(1+r)^i} \frac{1}{(1+\Lambda)^i} + \frac{1}{(1+r)^n} \frac{1}{(1+\Lambda)^n} \\ &\quad + R \sum_{i=1}^n \frac{1}{(1+r)^i} \frac{\Lambda}{(1+\Lambda)^i} \end{aligned}$$

DEFAULT PROBABILITY AND RECOVERY VALUE

In fact, for any uncollateralized bond, there is an approximate relationship between recovery value and default probability. Consider a par floater paying an annual coupon of $r+S$; what should be the coupon spread S ? Suppose the recovery value of the floater is R and default probability is λ , then the price of the par floater is

$$1 = \sum_{i=1}^n (r+S) \frac{1}{(1+r)^i} \frac{1}{(1+\Lambda)^i} + \frac{1}{(1+r)^n} \frac{1}{(1+\Lambda)^n} + R \sum_{i=1}^n \frac{1}{(1+r)^i} \frac{\Lambda}{(1+\Lambda)^i}$$

Rewriting the expression above

$$1 = \underbrace{\sum_{i=1}^n (r+\Lambda+r\Lambda) \frac{1}{(1+r)^i} \frac{1}{(1+\Lambda)^i}}_{=1} + \frac{1}{(1+r)^n} \frac{1}{(1+\Lambda)^n} + (S-\Lambda+R\Lambda-r\Lambda) \sum_{i=1}^n \frac{1}{(1+r)^i} \frac{1}{(1+\Lambda)^i}$$

Upon cancellation of 1 on both sides, we obtain the coupon spread

$$S = \Lambda(1-R+r) \approx \lambda(1-R+r)$$

Namely, the par-floater coupon spread equals to default probability multiplied by one minus recovery value plus interest rate.

For any uncollateralized bond, we can calculate its probability of default from its market price and a recovery value assumption. The market price of the bond is a market observable. The recovery value and default probability are not. We can then imagine the existence of a floater priced at par with the same maturity date and the same coupon payment date, and is subject to the same default probability and has the same recovery value as the uncollateralized bond, whatever they may be. Then we can take the attitude that the coupon spread of this par floater is a market observable, which should be invariant under recovery value assumptions. As such, the relationship above implies that for each uncollateralized bond, default probability multiplied by one minus recovery value plus interest rate is a constant. Indeed, numerical simulation shows this relationship is a good approximation in most cases.

where

$$\Lambda = e^{\lambda} - 1 \approx \lambda$$

This expression for price is simply the present value of the cash flows weighted by the probabilities of receiving them. Here, for simplicity, the interest rate zero curve is assumed to be flat at r .

It is interesting to note that if we set recovery value to zero, then the price of the bond is symmetric with respect to interest rate r and default probability Λ , which is approximately equal to λ for small λ . Therefore, interest rate duration (convexity) and default probability duration (convexity) should be approximately the same when recovery value is zero¹. Similarly, for floating rate bonds, the price is symmetric with respect to the yield of the bond and default probability when recovery value is zero².

Given the market price of a bond and recovery value R , this equation can be solved for λ , which is the independent static default probability of the bond if the interest rate r in future periods are assumed to be the forward rates implied by current term structure of the interest rates. For each uncollateralized bond, default probability is approximately related to the recovery value R through the following relationship³:

$$S = \Lambda(1 - R + r) \approx \lambda(1 - R + r)$$

where S is the coupon spread for a par-valued floater. In other words, default probability multiplied by one minus recovery value plus interest rate is a constant for all uncollateralized bonds. Indeed, Exhibit 5 in the main text shows this relationship is a very good approximation.

SIMULATE DEFAULT TIME FROM DEFAULT PROBABILITY PATH

If the path of default probability is flat – corresponding to the case of constant probability of default, then the time of default can be simulated by x/λ , where x is a random draw from a standard exponential distribution. For a step-wise path – corresponding to the case of dynamic probability of default, the time of default can be found by applying the simulation above repeatedly. Specifically, starting from λ_1 , the parameter governing the first (or current) time period (from 0 to t_1), we can simulate the time of default by x_1/λ_1 , if this is earlier than t_1 , then this is the time of default for this path. Otherwise, we have to simulate the time of default by $t_1 + x_2/\lambda_2$, where x_1 and x_2 are all random draws from a standard exponential distribution. This process will be repeated until the time of default falls into time interval from t_{i-1} to t_i governed by the λ_i .

For a dynamic default process, we allow the probability of default to change over time according to a term structure of volatility. We start from a chosen value of default probability, which is assumed to be the probability governing the current period. Then we simulate a path of default probabilities, allowing the probability to go up or down in future periods. We can then simulate the time of default dictated by this path.

We subsequently combine each time of default with a path of interest rate derived from a two-factor Heath-Jarrow-Morton process calibrated to interest rate swaptions and caps. Each pair of default time and interest rate path leads to a price of the bond under consideration. By nesting the entire process in a root finding routine, we can find the initial probability of default (for the current period, which is also the average probability of default for all the paths during future periods) such that the average price from all pairs hits the market price. This initial, or average, probability is the independent dynamic probability of default under a dynamic default process. As in the case of static default process, this dynamic probability of default can be bootstrapped by allowing the diffusion process of default probability to have discontinuous jumps for all its paths.

As mentioned in the main text, it is possible for us to price the value of repo specialness if we can make reasonable assumptions about repo rates in the future. The repo effect depends on the realized price of the bond. As such, a correct evaluation of the repo effect requires the appropriate handling of the path dependency on realized prices. Fortunately, the simulation approach adopted here has no problem dealing with path dependency. However, the challenge lies in the pricing of American options with simulation, which are what the options embedded in the callable Bradys are in essence.

1. This can be clearly seen for continuously compounded interest rates. In this case, the price of the bond is

$$P = \sum_{t=1}^n C e^{-ir} e^{-i\lambda} + e^{-nr} e^{-n\lambda}$$

The derivative of price with respect to r or λ is the same.

2. Here yield means the substitution of the term structure of risk free interest rate r by a constant yield y in the denominator in the expression for bond price. The default probability is still present in the expression for the bond price. As such, here the term yield is not exactly used in the same sense as the yield to maturity for a risky bond.

3. Sometimes this approximation has to be replaced by numerical calculation.

The way we approach this problem is to apply regressions to the bond prices during the backward induction along the simulation paths. The entire simulation is driven by five state variables, four from the two-factor interest rate process⁴ and one from the default process, which is simply the default probability λ . The four state variables from the interest rate process at each node of the simulation path completely determine the entire zero curve at that node. The value of λ controls the likelihood of default.

To price American options with simulation, we must compare the strike with the *expected* price at each node along each path. This *expected* price should incorporate all future call/non-call decisions. This *expected* price can be estimated by regressing the value of the bond *realized* on each path against the five state variables during the backward induction. The rationale behind this regression is that these state variables determine the *expected* value of the bond, as opposed to the *realized* value from a particular path in the future.

Finally, the term structure of default probability volatility can be derived from historical price volatilities. Price volatility can be viewed as coming from interest rate volatility and default probability volatility:

$$\frac{\Delta P}{P} = -D_{\Lambda}(R)\Lambda\left(\frac{\Delta\Lambda}{\Lambda}\right) - D_r(R)r\left(\frac{\Delta r}{r}\right)$$

where $D_{\Lambda}(R)$ and $D_r(R)$ are default probability duration and interest rate duration under recovery value R (which is 20% in our case), respectively.

Under the assumption of zero recovery value, interest rate duration approximately equals default probability duration for fixed coupon bonds.

$$D_{\Lambda}(0) = D_r(0)$$

On the other hand, the following relationship linking default probability under recovery value R and recovery value zero relates default probability durations under these two recovery value assumptions.

$$\Lambda(R) = \frac{1+r}{1-R+r}\Lambda(0) \Rightarrow D_{\Lambda}(R) = \frac{1-R+r}{1+r}D_{\Lambda}(0)$$

As such,

$$D_{\Lambda}(R) = \frac{1-R+r}{1+r}D_r(0)$$

In fact, the above relationship between default probability duration and interest rate duration holds approximately for default probability convexity and interest rate convexity as well. Thus, a positive interest rate convexity gives a positive default probability convexity as long as R is not too large.

Here we make an assumption that interest rate duration is not very sensitive to recovery value. Namely, we assume that $D_r(R) = D_r(0) = D_r$, the interest rate duration in the normal sense for a risky bond in emerging markets⁵. Under this assumption, the relationships above provide a link between default probability volatility and the price volatility:

$$\sigma_{\Lambda}^2 = \frac{\sigma_P^2 - D_r^2 r^2 \sigma_r^2}{D_r^2 \left(\frac{1-R+r}{1+r}\right)^2 \Lambda^2} = \frac{\sigma_P^2 - D_r^2 r^2 \sigma_r^2}{D_r^2 \left(\frac{S}{1+r}\right)^2}$$

In obtaining the above expression for default probability volatility, we have assumed that interest rate movements have zero correlation with default probability movements. Positive (negative) correlation between the two will reduce (increase) the default probability volatility.

The equation above is also valid for floating rate bonds with the substitution of yield duration for interest rate duration due to the fact that the price of a floater is symmetric with respect to yield and default probability under zero recovery assumption. All other arguments and assumptions are completely parallel to the case of fixed coupon bonds. Since the yield convexity for a floater is still positive, similar to the case of fixed coupon bonds, this symmetry leads to the positive default probability convexity.

In fact, all the inputs on the right hand side are market observable: price volatility, interest rate volatility, interest rate level, and par floater coupon spread. However, in the absence of a par floater with the same coupon payment dates and maturity date of the bond under consideration, we still need recovery value and default probability, instead of par floater coupon spread, as inputs.

As the first step to apply the equation above, we have to insert the default probability without volatility into the equation since we do not yet know the default probability volatility. We can then obtain a default probability volatility from the equation. However, the convexity effect due to this volatility will lead to a default

4. Ritchken, P., and L. Sankarasubramanian, 1995, "Volatility structures of Forward Rates and the Dynamics of the Term Structure." *Mathematical Finance* 5: 55-72

5. Usually $D_r(0)$ and D_r are very close. When the differences among these interest rate durations are not negligible, we account for their differences numerically.

probability larger than the zero volatility default probability – the default probability we begin with. In short, we need a default probability that is consistent with the volatility that is derived from it.

Since the increase in default probability is due to the convexity effect, if we start from the default probability under zero volatility, an increase in the implied default probability due to an introduction of volatility must be proportional to the volatility squared. In other words, we have (we ignored the minor contribution from interest rate convexity):

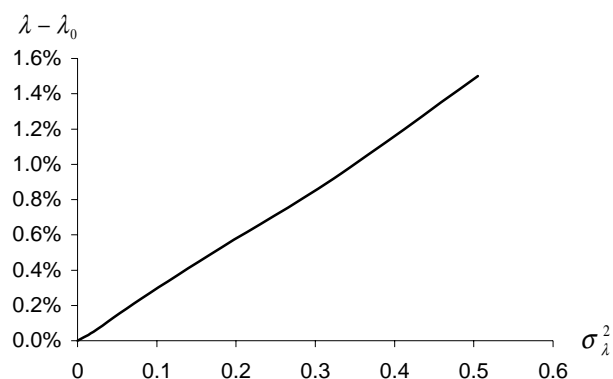
$$\lambda - \lambda_0 \propto \sigma_\lambda^2 \quad \text{or} \quad \lambda - \lambda_0 \approx a \sigma_\lambda^2$$

where λ_0 denotes the default probability under zero volatility and a denotes the proportionality factor. Exhibit 16 shows this is indeed the case.

Exhibit 16

Default prob changes linearly with volatility squared

Difference between default probability with a volatility and without volatility vs volatility squared (UMS 26)



Source: J.P. Morgan

The two equations above lead to a cubic algebraic equation for the self-consistent default probability, which has a unique real solution as long as λ_0 is positive, which is true for almost all cases.

The self-consistent default probability can be significantly larger than the zero volatility default probability. As such, finding this probability is a very important step and it has a significant impact on the volatility of default probability.

The volatility derived from this self-consistent default

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probability is the average volatility prevailing from now until the maturity of the bond. A set of such average volatilities from a set of bonds with different maturities can be bootstrapped into the forward volatilities – term structure of volatility. Assuming the default probabilities governing different time periods are not correlated, we can bootstrap the forward volatilities from the average volatilities through the following relationship:

$$\lambda_T^2 \sigma_{T-t}^2 (T-t) = \lambda_T^2 \sigma_T^2 T - \lambda_t^2 \sigma_t^2 t$$

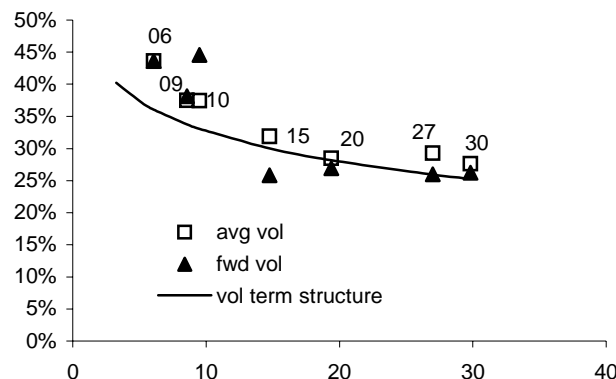
Here σ_{T-t} denotes the forward volatility between time T and t ($T > t$). λ_T and λ_t represents the self-consistent default probability until time T and t , respectively. Similarly, σ_T and σ_t denote the self-consistent average volatility until time T and t , respectively.

Given that bond prices are influenced by many technical factors, we would like a smooth term structure of volatility. We assume that volatility decreases with the logarithm of time since we have found that volatility generally drops much more sharply at the short end than the long end. In the end, we adjust the parameters such that the overall cumulative variance of default probabilities implied by the term structure equals to that implied by the longest bond under consideration. An example is shown in Exhibit 17 for the term structure of default probability volatility of Argentine Eurobonds.

Exhibit 17

Term structure of default probability volatility can be backed out from historical price volatilities

Default prob vol of Argentine Eurobonds vs. bond maturity (yrs)



Source: J.P. Morgan

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